Algebra Preliminary Exam 1

1. Let p be a prime and G be a p-group. Let $Z(G)$ be the center of G. Show that if N is a normal subgroup of G and $N \neq \{1\}$, then $Z(G) \cap N \neq \{1\}$.

2. Let $\mathbb{Q}(x)$ be the ring of rational functions with rational coefficients. Prove $\mathbb{Q}(x)(\sqrt{1-x^3})$ is isomorphic to $\mathbb{Q}(x)(\sqrt[3]{1-x^2})$.

3. Let $A, B \in M_n(\mathbb{C})$, the ring of $n \times n$ matrices with complex entries. Assume $A^n = B^n = 0$ and $AB = BA$. Prove $(A+B)^n = 0$.

4. Determine the commutator subgroup of D_{13} upto isomorphism.

5. Let $a(x) \in \mathbb{R}(x)$, the ring of rational functions with real coefficients. Does there exist a non-zero polynomial $p(t) \in \mathbb{R}[t]$ such that $p(a(x)) = 0$?

6. Let R be the ring $\mathbb{Z}[i]/(3)$ where (3) is the ideal generated by 3 in $\mathbb{Z}[i]$.

   a) Determine the group of ring automorphisms of R upto isomorphism.

   b) Determine the group of $\mathbb{Z}[i]$-module automorphisms of R upto isomorphism.