Algebra Preliminary Exam 2

1. Consider the group of invertible complex matrices $G = GL_3(\mathbb{C})$. Consider all $M \in G$ such that $M^5 = 4M$. How many conjugacy classes do such matrices lie in ?

2. Let R be a local Noetherian ring and M be a free R-module. Let $n > m$ be positive integers and let $\phi : M^n \to M^m$ be a surjective R-module homomorphism. Prove $ker(\phi)$ is also a free R-module?

3. Let α be a complex number whose minimal polynomial over $\mathbb{Q}$ has degree 5. Prove $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^2)$.

4. How many 7 degree monic irreducible polynomials are there in $\mathbb{F}_2[x]$?

5. Let $\mathbb{F}$ be a field and $V_1, V_2, \ldots, V_n$ be $\mathbb{F}$-modules. Assume the sequence

$$0 \to V_1 \to \ldots \to V_i \to \ldots \to V_n \to 0$$

is exact at every step. Prove

$$\sum_{i=1}^{n} (-1)^i \dim(V_i) = 0.$$

6. Determine all homomorphic images of A_4, the alternating group on 4 symbols.