Algebra Preliminary Exam 3

1. Prove that two finite abelian groups with the same number of elements of each order are isomorphic.

2. Let R be a nonzero commutative ring, A an $m \times n$ matrix with coefficients in R, and let $\phi : R^n \to R^m$ be the left multiplication by A. Prove that the following are equivalent:

   a) ϕ is surjective.

   b) A has a right inverse (*i.e.*there is a matrix B with coefficients in R such that $AB = I_m$.

   c) The determinants of $m \times m$ submatrices of A generate the unit ideal of R.

3. Let $A = (a_{ij})$ be any $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Prove that for each $i = 1, \ldots, n$, $\lambda_1 \geq |a_{ii}| \geq \lambda_n$.

4. Let M be an R-module over a commutative ring with unity. Let $f_1, \ldots, f_m \in \mathrm{Hom}(M,R)$ and let

$$N = \bigcap_{i=1}^{m} \ker f_i.$$

   Prove that M/N is finitely generated.

5. For which triples (n, p, k) of positive integers is $GL_n(\mathbb{Z}/p^k\mathbb{Z})$ a solvable group?

6. Prove that any projective module is a direct summand of a free module.