MTH 246 Lecture Notes

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Course Information

Topics: Algebraic Combinatorics and the Theory of Symmetric Functions

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Introduction/Motivation

Why study symmetric functions? They have a broad collection of applications. For example, in physics, the wave functions of bosons are symmetric functions. As we will see in this course, these are also very useful in combinatorics and representation theory.

The Ring of Symmetric Functions

Definition: For $x = (x_1, x_2, ...)$, a set of indeterminants, and $n \in \mathbb{N}$, a homogeneous *symmetric function* of degree n over a commutative ring R (with 1) is a formal power series $f(x) = \sum_{\alpha} c_\alpha x^\alpha$ where

(a) $\alpha = (\alpha_1, \alpha_2, ...)$ ranges over all compositions of n;
(b) $c_\alpha \in R$;
(c) $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ...$;
(d) $f(x_{\omega(1)}, x_{\omega(2)}, ...) = f(x_1, x_2, ...)$ for all $\omega \in S_\infty$.

Note that a symmetric function of degree 0 is an element in R.

Definition: Let Λ_R^n (or just Λ^n if R is understood by context) be the set of all homogeneous symmetric functions of degree n over R.

Remarks: (1) If $f, g \in \Lambda_R^n$ and $a, b \in R$, then $af + bg \in \Lambda_R^n$. Hence, Λ_R^n is an R-module which implies that for $R = \mathbb{Q}$, $\Lambda_{\mathbb{Q}}^n$ is a $\mathbb{Q}$-vector space. (2) If $f \in \Lambda_R^n$ and $g \in \Lambda_R^m$, then $f \cdot g \in \Lambda_R^{n+m}$.

These motivate the following definition:

Definition: Consider $\Lambda_R = \Lambda_R^0 \oplus \Lambda_R^1 \oplus \Lambda_R^2 \oplus ....$ Then Λ_R is the set of power series $f = f_0 + f_1 + f_2 + ...$ where $f_n \in \Lambda_R^n$ with all but finitely many $f_n = 0$. We call Λ_R the ring of symmetric functions. Note that Λ_R has a natural grading.

We want to study various bases for Λ_R. To do this, we use partitions to label each basis.

Partitions

Definition: A partition λ of $n \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, ..., \lambda_k) \in \mathbb{N}^k$ such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ and $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_k = n$. We also allow strings of zeros at the end, but we identify these with the partition without zeros.

Definition: Let P_n be the set of partitions of n and $P = \cup_{n \geq 0} P_n$.

Examples of P_n

Partitions of n	Ferrer's Diagrams		
$P_0 = \{\emptyset\}$			
$P_1 = \{1\}$	●		
$P_2 = \{2, 11\}$	● ●	● ●	
$P_3 = \{3, 21, 111\}$	● ● ●	● ● ●	● ● ●

Definition: We write $\lambda \vdash n$ if $\lambda \in P_n$. The length of λ, denoted $l(\lambda)$ is the number of nonzero parts of λ. (By parts we mean the λ_i's.) Finally, let $m_i(\lambda)$ be the number of parts of λ of size i.

Example(s):

$$\lambda = (5,3,3,1,1,1) = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & \\ \bullet & & & & \\ \bullet & & & & \\ \bullet & & & & \end{array}$$

For λ above, we have $l(\lambda) = 6$, $m_1(\lambda) = 3$, $m_2(\lambda) = 0$, $m_3(\lambda) = 2$, ...

Definition: We can also express any partition λ in frequency notation, given by $\lambda = < 1^{m_1} 2^{m_2} 3^{m_3} ... >$. In the example above, $(5,3,3,1,1,1) = < 1^3 3^2 5^1 >$. The transpose λ^t of a partition λ is the partition $\lambda^t = (\lambda_1' \lambda_2' \lambda_3' ...)$ where λ_i' is the length of the ith column of λ. In the example above, $\lambda^t = (6,3,3,1,1)$.

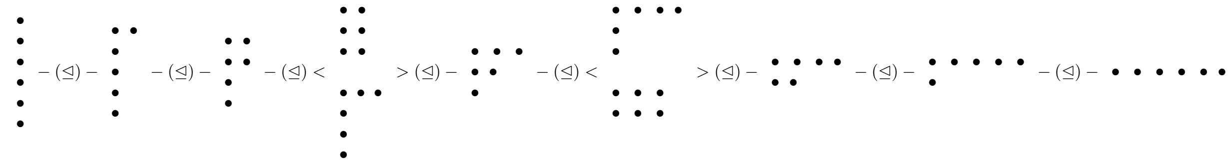
Definition: (Containment Order) For $\lambda, \mu \in P$, we say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. (Since μ and λ may be partitions of different natural numbers, it is clear now why we allow strings of zeros to be appended to a partition.) The pair $(P, \subseteq)$ forms a lattice, called the Young lattice Y. The rank of λ in Y is $|\lambda|$. From this, we obtain the rank generating function

$$F(Y, x) = \sum_{\lambda \in Y} x^{\text{rank}(\lambda)} = \prod_{i \geq 1} \frac{1}{1 - x^i}.$$

January 7, 2009 -

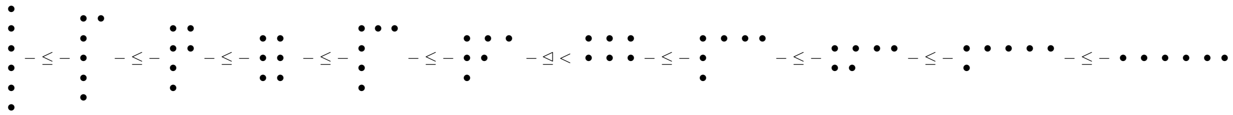
Definition: (Dominance Order) Let $\lambda, \mu \in P_n$. (So $|\mu| = |\lambda|$.) We write $\mu \trianglelefteq \lambda$ if $\mu_1 + ... + \mu_i \leq \lambda_1 + ... + \lambda_i$ for all i.

Example(s): For $n = 6$, we obtain the following partial order:



Next, we define a total order on the partitions.

Definition: (Lexicographic Order) For $\mu, \lambda \in P_n$, we say $\mu \leq \lambda$ if either $\mu = \lambda$ or
$\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$, ... $\lambda_i = \mu_i$, and $\lambda_{i+1} > \mu_{i+1}$.



Proposition: Lexicographic order is a refinement of dominance order. That is, if $\lambda, \mu \vdash n$, then $\mu \trianglelefteq \lambda \Rightarrow \mu \leq \lambda$.

Proof: If $\mu \neq \lambda$, find the smallest i such that $\mu_i \neq \lambda_i$. Then $\mu_1 + ...\mu_{i-1} = \lambda_1 + ... + \lambda_{i-1}$ and $\mu_1 + ... + \mu_i < \lambda_1 + ... + \lambda_i$ imply $\mu_i < \lambda_i$.

Monomial Symmetric Functions

Definition: For $\lambda \vdash n$, the monomial symmetric function $m_\lambda(x) \in \Lambda^n$ is defined as $m_\lambda(x) = \sum\limits_{\alpha} x^\alpha$ where α runs over all distinct permutations $\alpha = (\alpha_1, \alpha_2, ...)$ of $\lambda = (\lambda_1, \lambda_2, ...)$.

Example(s):

- $m_\emptyset = 1$,
- $m_1 = x_1 + x_2 + x_3 + ...,$
- $m_2 = x_1^2 + x_2^2 + x_3^2 + ...,$
- $m_{11} = x_1x_2 + x_1x_3 + x_1x_4 + ... + x_2x_3 + x_2x_4 + ... = \sum\limits_{i<j} x_ix_j = \frac{1}{2}(m_1^2 - m_2).$

We claim that $\{m_\lambda \mid \lambda \vdash n\}$ is a basis for Λ^n. If $f = \sum\limits_{\alpha} c_\alpha x^\alpha \in \Lambda^n$, then $f = \sum\limits_{\lambda \vdash n} c_\lambda m_\lambda$.
So, $\dim \Lambda^n = p(n) = |P_n|$, the number of partitions of n.

Elementary Symmetric Functions

Definition: The elementary symmetric functions e_λ for $\lambda \in P$ are defined by taking

$$e_n = m_{1^n} = m_{\underbrace{11...1}_{n \text{ times}}} = \sum_{i_1 < ... < i_n} x_{i_1} ... x_{i_n}$$

for $n \geq 1$ and forming $e_\lambda = e_{\lambda_1} e_{\lambda_2} ...$ for $\lambda = (\lambda_1, \lambda_2, ...) \in P$.

We wish to show that these form a basis for Λ^n as well. So, we relate them to the monomial symmetric functions.

Definition: Let $A = (a_{ij})_{i,j \geq 1}$ be an integer matrix with finitely many nonzero entries. We define the following: its row sum $r_i = \sum\limits_{j \geq 1} a_{ij}$, its column sum $c_j = \sum\limits_{i \geq 1} a_{ij}$, its row sum vector $\text{row}(A) = (r_1, r_2, ...)$, and its column sum vector $\text{col}(A) = (c_1, c_2, ...)$.

A $(0,1)$-matrix is such a matrix with $a_{ij} \in \{0, 1\}$ for all $i, j \geq 1$.

Proposition: For $\lambda \vdash n$ and $\alpha = (\alpha_1, \alpha_2, ...)$ a weak composition of n, the coefficient $M_{\lambda\alpha}$ of x^α in e_λ is the number of $(0,1)$-matrices $A = (a_{ij})_{i,j\geq 1}$ such that $\text{row}(A) = \lambda$ and $\text{col}(A) = \alpha$.

Proof: Let $X = \begin{bmatrix} x_1 & x_2 & x_3 & ... \\ x_1 & x_2 & x_3 & ... \\ ... & & & \end{bmatrix}$.

To get $e_\lambda = e_{\lambda_1} e_{\lambda_2} ...$ choose λ_1 terms from first row, λ_2 from the second row, and so on... such that the product of these is x^α. These chosen terms correspond to 1 and the others correspond to 0. Therefore, we obtain a $(0,1)$-matrix A with $\text{row}(A) = \lambda$ and $\text{col}(A) = \alpha$.

January 9, 2009 -

Corollary: $M_{\lambda\mu} = M_{\mu\lambda}$.

Proof: We obtain a bijection between the $(0,1)$-matrices of one side with the $(0,1)$-matrices of the otherside via the transpose. That is, A is a $(0,1)$-matrix with $\text{row}(A) = \lambda$ and $\text{col}(A) = \mu$ if and only if A^t is a $(0,1)$-matrix with $\text{col}(A^t) = \lambda$ and $\text{row}(A^t) = \mu$.

A combinatorial interpretation of this is that for n balls, λ_i balls labelled i, and boxes labelled $i \in \mathbb{N}_{>0}$, $M_{\lambda\mu}$ is the number of ways of placing balls into boxes such that (a) no box contains more than one ball of a given color; and (b) box i contains μ_i balls.

Definition: Let $\{u_\lambda\}$ be a basis of Λ and $f \in \Lambda$ where $f = \sum_{\lambda \in P} c_\lambda u_\lambda$ with $c_\lambda \in R$. We say f is u-positive if $c_\lambda \geq 0$ and f is u-integral if $c_\lambda \in \mathbb{Z}$.

Proposition: $\prod_{i,j\geq 1} (1 + x_i y_j) = \sum_{\lambda \in P} m_\lambda(x) e_\lambda(y)$.

Proof: The monomial $x^\alpha y^\beta = (x_1^{\alpha_1} x_2^{\alpha_2} ... y_1^{\beta_1} y_2^{\beta_2} ...)$ appearing in the expansion of $\prod_{i,j}(1 + x_i y_j)$ corresponds to the $(0,1)$-matrix $A = (a_{ij})$ with finitely many 1's satisfying $\prod_{i,j}(x_i y_j)^{a_{ij}} = x^{\text{row}(A)} y^{\text{col}(A)}$. This implies that the coefficients of $x^\alpha y^\beta$ in $\prod_{i,j}(1 + x_i y_j)$ equals the number of $(0,1)$-matrices A with $\text{row}(A) = \alpha$ and $\text{col}(A) = \beta$. This gives us that $\prod_{i,j}(1 + x_i y_j) = \sum_{\alpha,\beta} M_{\alpha\beta} x^\alpha y^\beta = \sum_{\lambda,\mu} M_{\lambda,\mu} m_\lambda(x) m_\mu(y) = \sum_\lambda m_\lambda(x) e_\lambda(y)$.

Theorem: For $\lambda, \mu \vdash n$, $M_{\lambda\mu} = 0$ unless $\mu \trianglelefteq \lambda^t$ and $M_{\lambda\lambda^t} = 1$. Hence $\{e_\lambda \mid \lambda \vdash n\}$ form a basis for Λ^n. Equivalently, $e_1, e_2, ...$ are algebraically independent and generate Λ as a $\mathbb{Q}$-algebra $\Lambda = \mathbb{Q}[e_1, e_2, ...]$.

Proof: Suppose $M_{\lambda\mu} \neq 0$. Then there exists a $(0,1)$-matrix A with $\text{row}(A) = \lambda$ and $\text{col}(A) = \mu$. Let A' be the $(0,1)$-matrix with $\text{row}(A') = \lambda$ and 1's left-justified. That is, A'_{ij} for $1 \leq j \leq \lambda_i$ and 0 everywhere else. For all i, the number of $1's$ in the first i columns of A' is greater than or equal to the number of rows in the first i columns of A. This means that $\lambda^t = \text{col}(A') \trianglerighteq \text{col}(A) = \mu$. In addition, A' is the only $(0,1)$-matrix with $\text{row}(A) = \lambda$ and $\text{col}(A) = \lambda^t$. So, $M_{\lambda\lambda^t} = 1$. This proves the first statement of the theorem.

Next, we choose an ordering of P_n, $\lambda^1 < \lambda^2 < ... < \lambda^{p(n)}$ that is compatible with dominance order such that the reverse conjugate order $(\lambda^{p(n)})^t, ..., (\lambda^1)^t$ is also compatible with dominance. (We can do this because, as we observed earlier, dominance order is symmetric with respect to the transposes.) Now, we have $M_{\lambda\mu}$ with row order $\lambda^1, \lambda^2, ...$ and column order $(\lambda^1)^t, (\lambda^2)^t, ...$, which is upper uni-triangular by the first part of the theorem. From this, it is clear that $M_{\lambda\mu}$ has determinant 1 and is invertible. Earlier, we wrote the e_λ's in terms of m_λ's with $M_{\lambda\mu}$. Since $M_{\lambda\mu}$ is invertible, we can write the m_λ's in terms of $e'_\lambda s$. So, $\{e_\lambda \mid \lambda \vdash n\}$ form a basis for Λ^n. In fact, $\Lambda^n_{\mathbb{Z}}$ since the diagonal entries are 1.

Example(s):

- $e_{11} = e_1 e_1 = (x_1 + x_2 + ...)^2 = \sum_{i,j} x_i x_j = \sum_i x_i^2 + 2\sum_{i<j} x_i x_j = m_2 + 2m_{11}$.
- $e_{21} = e_2 e_1 = (\sum_{i<j} x_i x_j)(\sum_k x_k) = \sum_{i<j,k} x_i x_j x_k = \sum_{\substack{i\,<\,j \\ k\,=\,i \text{ or } k\,=\,j}} x_i x_j x_k + 3\sum_{i<j<k} x_i x_j x_k = m_{21} + 3m_{11}$