MAT 21B: CALCULUS FINAL EXAMINATION

THURSDAY, MARCH 22, 2001. 1:30 - 3:30 P.M.
ROOM: **198 YOUNG**
INSTRUCTOR: M. MULASE

Name: (Last) ____________ (First) ____________

Student ID Number: _____ – _____ – _____

CRN Number: ____________

Signature: ____________

Remark.

1. The test consists of 6 pages, including the cover sheet.
2. **Do not de-staple the test.**
3. **No calculators are allowed.**
4. It is an open-book, open-notes exam.
5. Scratch papers are allowed.
6. You have full two hours.
7. If you have any questions, ask the proctor before it becomes too late!

Scores:

Page 2: __________/11	Page 5: __________/4
Page 3: __________/8	Page 6: __________/11
Page 4: __________/6	

Total: __________/40

Problem 1 (11 points)**.** Evaluate the following integrals:

$$\int_1^e \ln x dx = (x\ln(x) - x)\Big|_1^e = 1. \tag{1}$$

$$\int_{-\pi/2}^{\pi/2} \cos x dx = \sin(x)\Big|_{-\pi/2}^{\pi/2} = 1 + 1 = 2. \tag{2}$$

$$\int_{-\sqrt{5}}^{\sqrt{5}} \sin(60x) dx = 0. \tag{3}$$

$$\int_0^{\pi} \cos(10x) dx = 0. \tag{4}$$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x)\Big|_{-1}^{1} = \frac{\pi}{2} - \frac{-\pi}{2} = \pi. \tag{5}$$

$$\int_{-1}^{1} \sqrt{1-x^2} dx = \pi/2. \text{ (The area of a hemi-circle.)} \tag{6}$$

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \tan^{-1}(x)\Big|_0^{\infty} = \frac{\pi}{2}. \tag{7}$$

$$\begin{aligned} \int_0^{\infty} x^2 e^{-x} dx &= -x^2 e^{-x}\Big|_0^{\infty} + 2\int_0^{\infty} x e^{-x} dx \\ &= (-x^2 e^{-x} - 2xe^{-x} - 2e^{-x})\Big|_0^{\infty} = 2. \end{aligned} \tag{8}$$

$$\int_{-2}^{2} x^{101} dx = 0. \tag{9}$$

$$[n > 0] \qquad \int_0^{\infty} e^{-nx} dx = -\frac{1}{n} e^{-nx}\Big|_0^{\infty} = \frac{1}{n}. \tag{10}$$

$$[n \geq 2] \qquad \int_1^{\infty} \frac{1}{x^n} dx = \frac{1}{1-n} x^{1-n}\Big|_1^{\infty} = \frac{1}{n-1}. \tag{11}$$

Problem 2 (8 points)**.** Let us consider a curve given by a parametric equation

$$\begin{cases} x = \cos\left(2\pi(1-e^{-t})\right) \\ y = \sin\left(2\pi(1-e^{-t})\right) \end{cases},$$

where $0 \le t < \infty$.

1. Find the equation of the curve in the Cartesian coordinate and give it in its simplest form:

$$x^2 + y^2 = 1.$$

2. Find the speed of the particle on the curve determined by the above equations at time t.

(a) Work: $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2} =$

$$\sqrt{(2\pi)^2(e^{-t})^2\sin^2\left(2\pi(1-e^{-t})\right) + (2\pi)^2(e^{-t})^2\cos^2\left(2\pi(1-e^{-t})\right)}.$$

(b) Answer in its simplest form:

$$2\pi e^{-t}.$$

3. What is the arc length of the curve when t varies from 0 to $+\infty$?

(a) Formula:

$$\int_0^\infty 2\pi e^{-t}dt.$$

(b) Work:

$$\int_0^\infty 2\pi e^{-t}dt = -2\pi e^{-t}\Big|_0^\infty = 2\pi.$$

(c) Answer in its simplest form:

$$2\pi.$$

4. (2 points) What is this curve? Give the most precise description.

A circle of redius 1 centered at the origin. (If no mention about the radius, subtract 1 point.)

Problem 3 (6 points)**.** Consider the plane region R that is bounded by the graph of $y = \sin(x)$ and the x-axis for $0 \leq x \leq \pi$ (see Figure 2). Let $(\overline{x}, \overline{y})$ be the centroid of R.

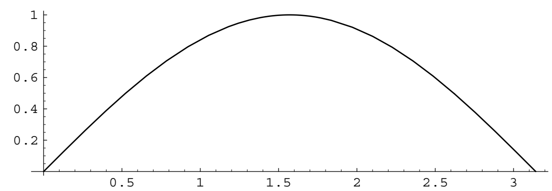

FIGURE 1. The plane region R.

1. Find the x-coordinate of the centroid.
   (a) Formula:
   $$\int_0^\pi x\sin(x)dx \Big/ \int_0^\pi \sin(x)dx.$$
   (b) Give your answer in its simplest form: $\overline{x} = \pi/2$.

2. Find the y-coordinate of the centroid.
   (a) Formula:
   $$\frac{1}{2}\int_0^\pi \sin^2(x)dx \Big/ \int_0^\pi \sin(x)dx.$$
   (b) work:
   $$\frac{1}{2}\int_0^\pi \sin^2(x)dx = \frac{1}{2}\int_0^\pi \frac{1-\cos(2x)}{2}dx = \frac{\pi}{4}.$$
   (c) Give your answer in its simplest form: $\overline{y} = \pi/8$.

3. Let V_x be the volume of the solid of revolution that is obtained by rotating R about the x-axis, and V_y the volume of the solid of revolution of R about the y-axis. Which of these volumes is larger? Answer: ____V_y____ is larger than the other.
   $$V_x = 2\pi \cdot \pi/8 \cdot 2, \qquad V_y = 2\pi \cdot \pi/2 \cdot 2.$$

Problem 4 (15 points)**.** Let us consider a 12-petal rose defined by $r = \cos(6\theta)$, for $0 \le \theta < 2\pi$. (see Figure 1).

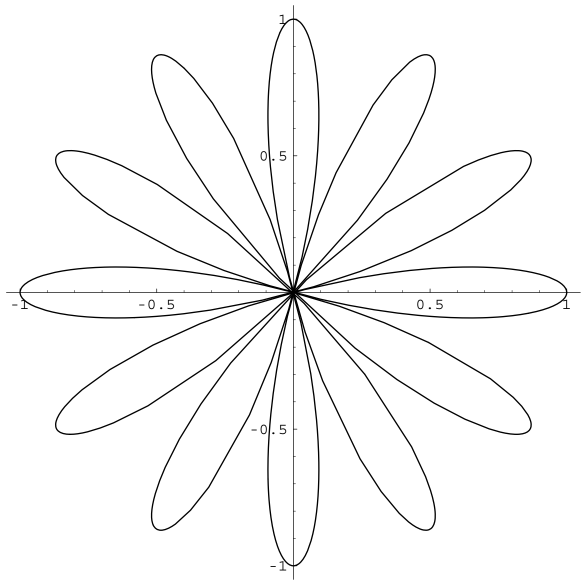

FIGURE 2. A 12-petal rose defined by $r = \cos(6\theta)$, $0 \le \theta < 2\pi$.

1. The formula for the *total area* of the region inside the 12-petal rose is:

$$\text{The Total Area} = \int_0^{2\pi} \frac{1}{2}\cos^2(6\theta)d\theta.$$

2. (2 points) Work: compute the integral.

$$\int_0^{2\pi} \frac{1}{2}\cos^2(6\theta)d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1+\cos(12\theta)}{2}d\theta$$
$$= \frac{1}{2}\cdot\frac{2\pi}{2} = \frac{\pi}{2}.$$

3. Answer: The total area of the region of the 12-petal rose is $\underline{\quad \pi/2 \quad}$.

4. As the angle θ changes $0 \le \theta < 2\pi$, how many times the curve passes through the origin? Answer: ___12___ times.
5. (2 points) Give a parametric equation for the 12-petal rose in terms of a single parameter θ.
$$\begin{cases} x = \cos(6\theta) \cdot \cos(\theta) \\ y = \cos(6\theta) \cdot \sin(\theta) \end{cases} .$$
6. As the angle θ changes $0 \le \theta < 2\pi$, how many times the curve passes through the point $(1, 0)$ in the Cartesian coordinate? Answer: ___once___.
7. How about the total area of the region inside the curve defined by $r = \cos(600\theta)$ for $0 \le \theta < 2\pi$? The area is ___$\pi/2$___.
8. (3 points) Next let us consider a new curve defined by $r = \cos(\theta)$, $0 \le \theta < 2\pi$. Describe this curve. Give the best possible answer you can give.

   The curve is a circle of radius $1/2$ centered at the point $(1/2, 0)$ in the Cartesian coordinate. Or equivalently, it is a circle of radius $1/2$ that passes through the origin and $(1, 0)$. 1 pt for circle, 1 pt for radius, and 1 pt for other information.
9. When the angle changes from 0 to 2π, how many times the point (r, θ) of the polar coordinate traces around the curve $r = \cos(\theta)$? Answer: ___twice___.
10. Compute the area of the region inside the curve $r = \cos(\theta)$. Answer: ___$\pi/4$___.
11. Considering θ being time, what is the speed of a particle on the curve $r = \cos(\theta)$ as θ moves from 0 to 2π? Answer: ___1___.
$$\sqrt{r^2 + (r')^2} = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$