# MAT 145: Homework Solutions #1

Prepared by Maya Ahmed

April 3, 2003

1. **Brualdi 2.4**

   Show that if $n+1$ integers are chosen from the set $\{1, 2, 3, .., 2n\}$, then there are always two which differ by 1.

   **Answer:**

   Partition the set $\{1, 2, 3, .., 2n\}$ into $n$ subsets as following

   $\{1, 2\}\{3, 4\}, ...., \{2n-1, 2n\}$. Now if we choose $n+1$ distinct numbers from $\{1, 2, 3, .., 2n\}$, then two must come from the same subset by pigeonhole principle. These two numbers must then differ only by 1.

2. **Brualdi 2.5**

   Show that if $n+1$ integers are chosen from the set $\{1, 2, 3, .., 3n\}$, then there are always two which differ by at most two.

   **Answer:**

   Partition the set $\{1, 2, 3, .., 3n\}$ into $n$ subsets as following

   $\{1, 2, 3\}, \{4, 5, 6\}, ...., \{3n-2, 3n-1, 3n\}$. Now if we choose $n+1$ distinct numbers from $\{1, 2, 3, .., 3n\}$, then two must come from the same subset by pigeonhole principle. These two numbers must then differ by no more than two.

3. **Brualdi 2.8**

   Use the pigeonhole principle to prove that the decimal expansion expansion of a rational number $m/n$ eventually is repeating.

   **Answer:**

   Assume $m < n$, since if $m > n$ then after dividing $m$ by $n$ we get $m = pn + r$ and the proof will apply to $r/n$.

   Let $m/n = .a_1a_2a_3....$ , i.e

   $$\frac{m}{n} = \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + .....$$

If we consider the steps of long division of $m$ by $n$, what is happening is :

$$\begin{aligned}\frac{m}{n} &= \frac{r_0}{n}\\ &= \frac{a_1}{10^1} + \frac{r_1}{n10}\\ &= \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{r_2}{n10^2}\\ &....\end{aligned}$$

Where $r_i$ is the remainder after $i$ divisions.

Note that

$$\frac{r_i}{n10^i} = \frac{a_{i+1}}{10^{i+1}} + \frac{r_{i+1}}{n10^{i+1}}$$

Therefore $a_{i+1}$ and $r_{i+1}$ are uniquely defined by $r_i$. So if $r_i = r_{i+k}$ then $r_{i+1} = r_{i+k+1}$ and $a_{i+1} = a_{i+k+1}$ and we see that the decimal expansion repeats after every $k$ steps.

Therefore if we show $r_i = r_{i+k}$ for some $i, k$ we are done. But we know that $r_i \in \{0, 1, ..., n-1\}$ for $i = 0, 1, 2, ..., n$ because $r_i$ is the remainder when a integer is divided by $n$. Since we have $n+1$ remainders, by pigeonhole principle some $r_i = r_j$. Assume $i < j$. Then $k = j - i$. And that proves the result.

4. **Brualdi 2.12**

   Show by example that the conclusion of the Chinese remainder theorem need not hold when $m$ and $n$ are not relatively prime.

   **Answer:**

   Let $m = 2, n = 4, a = 1, b = 2$. $m, n$ are not relatively prime. If there was an number such that $x = 2p + 1$ and $x = 4q + 2$ then $x$ is both odd and even. This is not possible. Hence the result.

5. **Brualdi 2.14** A bag contains 100 apples, 100 bananas, 100 oranges and 100 pears. If I pick one piece of fruit out of the bag every minute, how long will it be before I am assured of having at least a dozen pieces of fruit of the same kind?

   **Answer:**

   Let $q_1$ be the number of apples picked. Let $q_2$ be the number of bananas picked. Let $q_3$ be the number of oranges picked. Let $q_4$ be the number of pears picked.

   We are interested in the case $q_1 = q_1 = q_3 = q_4 = 12$. By the strong form of the pigeon hole principle, we need to pick $12 + 12 + 12 + 12 - 4 + 1 = 45$ fruits before we are assured that we have a dozen of each kind.

6. **Brualdi 2.15**

   Prove that for any $n + 1$ integers $a_1, a_2, \ldots a_{n+1}$ there exist two of the integers $a_i$ and $a_j$ with $i \neq j$ such that $a_i - a_j$ is divisible by $n$.

**Answer:**

For each $a_i$, we can write $a_i = q_i n + r_i$ where $0 \leq r_i \leq n-1$. Then we have $r_1, r_2, \ldots r_{n+1} \in \{0, 1, 2, \ldots n-1\}$. By the pigeon principle $r_i = r_j$ for some $i$ and $j$. Then

$$a_i - a_j = (q_i n + r_i) - (q_j n + r_j) = n(q_i - q_j)$$

Therefore $a_i - a_j$ is divisible by $n$.

7. **Brualdi 2.16**

   Prove that in a group of $n > 1$ people there are two who have the same number of acquaintances in the group. (It is assumed that no one is acquainted with him or herself.)

   **Answer:**

   Let $a_i$ be the number of acquaintances of the $i$ the person. Then $a_1, a_2, \ldots a_n \in \{0, 1, 2, \ldots n-1\}$. Proof is by contradiction. Suppose that everyone had a different number of acquaintances, then one person has 1 acquaintance, the other 2 acquaintance ... and one person has $n-1$ acquaintances. Now if one person has 0 acquaintance, then how can another have $n-1$ acquaintances? Hence here are two who have the same number of acquaintances in the group.

8. **Brualdi 2.17**

   There are 100 people at a party. Each person has an even number (possibly zero) of acquaintances. Prove that there are three people at the party with the same number of acquaintances.

   **Answer:**

   Let $a_i$ be the number of acquaintances of the $i$ the person. Then $a_1, a_2, \ldots a_n \in S = \{0, 2, 4, 6, \ldots 98\}$. The set $S$ has 50 elements. Proof by contradiction. Suppose there are no three people with the same number of acquaintances. Then exactly two people have same number of acquaintances. So 2 people have 0 acquaintances, another 2 people have 2 acquaintances, ..., another 2 people have 98 acquaintances. This is not possible. If two people have zero acquaintances, another two people cannot have 98 acquaintances. We are done.

9. **Brualdi 2.18**

   Prove that of any five points chosen within a square of side length 2, there are two whose distance apart is at most $\sqrt{2}$.

   **Answer:**

   Divide the square into 4 squares of side length 1. By pigeon hole principle since we have 5 points, two will lie in the same smaller square. Since the side length 1, the greatest distance between two points in the small square is the length of the diagonal $\sqrt{2}$.

10. **Brualdi 2.27**

A collection of subsets of $\{1, 2, \ldots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most $2^{n-1}$ subsets in the collection.

**Answer:**

Since a set and its complement cannot belong to this collection, and since the total number of subsets of any set is $2^n$, we get that there are at most $\frac{2^n}{2} = 2^{n-1}$ subsets in the collection.