

# MAT 145: Homework Solutions #2

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## 1. Brualdi 3.3

In how many ways can a poker hand (5 cards) be dealt? How many different poker hands are there?

**Answer:**

There are a total of 52 cards, so there are 52 choices for the first card. After the first card is chosen there are 51 choices for the second card. Similarly there are 50 cards for the 3rd card, 49 choices for the 4th card and 48 choices for the 5th card. Hence there are  $52 \times 51 \times 50 \times 49 \times 48$  poker hands.

Any five cards of the 52 cards can form a poker hand, hence there are  $\binom{52}{5}$  different poker hands.

## 2. Brualdi 3.5

Determine the largest power of 10 that is a factor of the following numbers (equivalently, the number of terminal 0's, using ordinary base 10 representation):

(a) 50!

**Answer:**

Since  $10 = 2 \cdot 5$ , we will search for factors of 50! that are divisible by 5 and 2.

Among all integers between 1 and 50, multiples of 5 appear once every five numbers. Thus the total number of the multiples of 5 in this range is  $50/5 = 10$ . All these 10 multiples of 5 contribute at least one factor of 5 in 50!. Now, we notice that 25 has *two* factors of 5. And 25 appears twice ( $50/25 = 2$ ) in the range. Thus we have to include two more factors of 5 in our count. Altogether, we have

$$\frac{50}{5} + \frac{50}{25} = 12$$

factors of 5. Since every other number is even, we immediately know there are more than  $50/2 = 25$  factors of 2 in 50!. Therefore, we conclude that there are exactly 12 powers of  $10 = 2 \cdot 5$  in 50!.

(b) 1000!

**Answer:**

Since multiples of 5 appear every five numbers, there are  $1000/5 = 200$  multiples of 5 in the range from 1 through 1000. Every multiple of 5 in this range contributes at least one factor of 5 in  $1000!$ . As before, we know that every multiple of 25 contributes at least two factors of 5. So we have to take into account  $1000/25 = 40$  more factors of 5 in  $1000!$ , because already one factor of 5 has been counted. (To avoid double counting, we add the number of multiples of 25 instead of two times of that.)

Actually, every multiple of  $125 = 5^3$  has at least *three* factors 5. Thus we have to add yet  $1000/125 = 8$  more factors into account. Finally, the largest power of 5 that does not exceed 1000 is  $625 = 5^4$ , so we need to take one more factor of 5 in our computation. Altogether, there are

$$\frac{1000}{5} + \frac{1000}{25} + \frac{1000}{125} + 1 = 249$$

factors of 5 in  $1000!$ .

Clearly every even number contributes at least one factor of 2, so  $1000!$  contains at least 500 factors of 2. Thus the total number of factors of  $10 = 2 \cdot 5$  is exactly 249.

**3. Brualdi 3.7**

In how many ways can six men and six ladies be seated at a round table if the men and ladies are to sit in alternate seats?

**Answer:** Let the six men be  $A_1, A_2, A_3, A_4, A_5, A_6$ . After seating  $A_1$  at the “head of the table” (see page 59 of text book), there are  $5!$  ways to seat the other 5 men, and  $6!$  ways to seat the six women such that the men and women are in alternate seats. So there are a total of  $5!6!$  ways of seating six men and six ladies at a round table if the men and ladies are to sit in alternate seats.

**4. Brualdi 3.8**

In how many ways can 15 people be seated at a round table if B refuses to sit next A? What if B only refuses to sit on A’s right?

**Answer:**

The total number of ways 15 people can sit around a round table is  $(15 - 1)! = 14!$  (see Theorem 3.2.2).

Now consider the arrangements where we make B sit to the right of A. Then there are thirteen more seats to fill and hence giving us a total of  $13!$  ways of doing this. So the number of ways 15 people can be seated at a round table if B only refuses to sit on A’s right is  $14! - 13! = 13 \cdot 13!$ . this answers the second part of the question.

Now we consider arrangements where B is sitting to the left of A and there are  $13!$  ways of doing this. So the number of ways 15 people can be seated at a round table if B refuses to sit next A (both to the right or left of A, talk of hating someone!) is  $13 \cdot 13! - 13! = 12 \cdot 13!$ .

5. **Brualdi 3.15**

Prove that

$$\binom{n}{r} = \binom{n}{n-r}$$

by using a combinatorial argument and not the values of these numbers as given in Theorem 3.3.1.

**Answer:**

The number of ways of choosing a set  $S$  of  $r$  elements from  $n$  elements is the same as choosing the complement set  $S^c$  of  $n - r$  elements. We know that  $\binom{n}{r}$  is the number of ways of choosing a set of  $r$  elements from  $n$  elements. Hence the proof.

6. **Brualdi 3.17** In how many ways can 2 red and 4 blue rooks be placed on an 8 by 8 board so that no two rooks can attack one another?

First consider the case when we have 6 indistinguishable rooks on the 8 by 8 chessboard so that they are not attacking each other. Since every row and column has at most one rook, there are 6 rows and 6 columns where you find a rook. The number of choices of these rows and columns is  $\binom{8}{6}^2$ . After you have chosen 6 rows and 6 columns, then you have reduced the size of the board effectively to 6 by 6, hence the number of non-attacking rook configurations, or the number of permutation matrices, is  $6!$ . Now we have to choose 2 out of 6 rooks to color them in red. The rest are colored in blue. The number of this choice is  $\binom{6}{2}$ . Altogether, the total number is  $\binom{8}{6}^2 \cdot 6! \cdot \binom{6}{2}$ .

7. **Brualdi 3.20** In how many ways can 5 indistinguishable rooks be placed on a  $8 \times 8$  chess board so that no rook can attack another and neither the first row nor the first column is empty?

**Answer:** Let  $(i, j)$  denote the coordinates of the chess board, where  $i$  is the row number and  $j$  is the column number. Place a rook in the  $(1, 1)$  cell. Then we cannot choose the first row or column, if the rooks are to be non-attacking. From the remaining 7 rows, we can choose  $\binom{7}{4}$  rows to place our rooks. Similarly we can choose  $\binom{7}{4}$  columns to place our rooks. Now for the first chosen row, we have 4 choices from the chosen columns to place our rooks. For the second row, we have 3 choices of columns, for the third row we have two choices of columns and the last chosen row we have only choice of columns to place our rooks. So in the case we place a rook in  $(1, 1)$  square we have  $\binom{7}{4}^2 4!$  ways to place 5 rooks such that they do not attack each other.

Now we consider the case when there is no rook in the  $(1, 1)$  square of the chess board. Then we have a choice of 7 squares to place a rook in the first row. Similarly we have 7 choices for placing a rook in the first column. We can thus place two rooks in the first row and first column such that they would not attack each other in  $7^2$  ways. Once we have placed rooks in the first row and first column, we have  $\binom{6}{3}$  choices of rows and  $\binom{6}{3}$  choices of columns to place the remaining 3 rooks. Like before, for the first chosen row, we have 3 chosen columns to place our rook, for the second row we have 2 choices

of column and so on. So for the case, when there is no rook in the  $(1, 1)$  square we have  $7^2 \binom{6}{3}^2 3!$  ways to place 5 rooks such that they do not attack each other.

Hence the total number of ways 5 indistinguishable rooks can be placed on a  $8 \times 8$  chess board so that no rook can attack another and neither the first row nor the first column is empty is  $\binom{7}{4}^2 4! + 7^2 \binom{6}{3}^2 3!$ .

### 8. Brualdi 3.26

Determine the number of 10-permutations of the multiset

$$S = \{3.a, 4.b, 5.c\}.$$

**Answer:**

(See example on page 70 for a similar problem.)

$$10! \left( \frac{1}{3!4!3!} + \frac{1}{3!3!4!} + \frac{1}{3!2!5!} + \frac{1}{2!4!4!} + \frac{1}{2!3!5!} + \frac{1}{1!4!5!} \right)$$

### 9. Brualdi 3.31

How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 30$$

satisfy  $x_1 \geq 2, x_2 \geq 0, x_3 \geq -5, x_4 \geq 8$ ?

**Answer:**

We need to reformulate this problem in new variables so that we are solving for variables greater than or equal to 0 (See example on page 74). So let  $y_1 = x_1 - 2, y_2 = x_2, y_3 = x_3 + 5, y_4 = x_4 - 8$ .

Then

$$y_1 + y_2 + y_3 + y_4 = x_1 - 2 + x_2 + x_3 + 5 + x_4 - 8 = x_1 + x_2 + x_3 + x_4 - 5$$

Thus we want to solve the equation  $y_1 + y_2 + y_3 + y_4 = 25$ , such that  $y_i \geq 0$  for all  $i$ .

Now we can apply Theorem 3.5.1 and we get that the number of nonnegative solutions of the equation

$$y_1 + y_2 + y_3 + y_4 = 25$$

is  $\binom{25+4-1}{25} = \binom{28}{25}$  which is also the number of integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 30$$

such that  $x_1 \geq 2, x_2 \geq 0, x_3 \geq -5, x_4 \geq 8$ .

10. **Brualdi 3.32**

There are twenty identical sticks lined up in a row occupying twenty distinct places:



and six of them are to be chosen.

(a) How many choices are there?

**Answer:**

There are  $\binom{20}{6}$  choices.

(b) How many choices are there if no two of the chosen sticks can be consecutive?

**Answer:**

Choose 6 sticks. Let the position of the chosen sticks be  $1 \leq r_1 < r_2 < r_3 < r_4 < r_5 < r_6 \leq 20$ . Now the remaining 14 sticks are between these six sticks. Let  $a_1$  be the number of sticks between  $r_1$  and 1, let  $a_2$  be the number of sticks between  $r_2$  and  $r_1$  and so on. Thus  $a_1 = r_1 - 1, a_2 = r_2 - r_1, a_3 = r_3 - r_2 \dots a_6 = r_6 - r_5, a_7 = 20 - r_6$ . Since we do not choose consecutive sticks,  $a_i \geq 1; i = 2, \dots, 6$  and  $a_1 \geq 0, a_7 \geq 0$ . Also  $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 14$ .

Now we make a change of variable (like problem 31). Let  $b_i = a_i - 1; i = 2, \dots, 6$ , let  $b_1 = a_1, b_7 = a_7$ . Then the required answer is the number of nonnegative integral solutions of the equation  $b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 = 9$ , which is  $\binom{9+7-1}{9} = \binom{15}{9}$ .

Thus there are  $\binom{15}{9}$  choices such that no two of the chosen sticks are consecutive.

(c) How many choices are there if there must be at-least two sticks between each pair of chosen sticks?

**Answer:**

This is similar to part (b), but we need two sticks between any pair of chosen sticks. So now  $a_i \geq 2, i = 2, \dots, 6$ . So we make the variable change  $b_i = a_i - 2; i = 2, \dots, 6$ , let  $b_1 = a_1, b_7 = a_7$ . Therefore the required answer is the number of nonnegative integral solutions of the equation  $b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 = 4$ . Thus the answer is  $\binom{4+7-1}{4} = \binom{10}{4}$ .