MAT 145: Homework Solutions #3

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1. **Brualdi 5.3** Consider the sum of the binomial coefficients along the diagonals of Pascal's triangle running upward from the left. The first few are: $1, 1, 1+1 = 2, 1+2 = 3, 1+3+1 = 5, 1+4+3 = 8$. Compute several more of these diagonal sums, and determine how these sums are related.

   Answer:

   Let D_i Denote the ith diagonal sum, that is, start from the ith row and go upward from the left. Then

$$D_i = \begin{cases} \binom{i}{0} + \binom{i-1}{1} + \binom{i-2}{2} + \cdots + \binom{\frac{i}{2}}{\frac{i}{2}} & \text{if n is even} \\ \\ \binom{i}{0} + \binom{i-1}{1} + \binom{i-2}{2} + \cdots + \binom{\frac{i+1}{2}}{\frac{i-1}{2}} & \text{if n is even} \end{cases}$$

   For example:

$$D_7 = \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3}$$

$$D_6 = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3}$$

$$D_5 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2}$$

$$D_4 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2}$$

   Claim: $D_i = D_{i-1} + D_{i-2}$.

   Proof:

   Case 1: i is even:

$D_i = \binom{i}{0} + \binom{i-1}{1} + \binom{i-2}{2} + \cdots + \binom{\frac{i}{2}}{\frac{i}{2}}$

Since $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, We get

$D_i = \binom{i}{0} + \{\binom{i-2}{1} + \binom{i-2}{0}\} + \{\binom{i-3}{2} + \binom{i-3}{1}\} + \cdots + +\{\binom{\frac{i}{2}+1}{\frac{i}{2}-1} + \binom{\frac{i}{2}+1}{\frac{i}{2}-2}\} + \binom{\frac{i}{2}}{\frac{i}{2}}$

Since $\binom{i}{0} = 1 = \binom{i-1}{0}$ and $\binom{\frac{i-2}{2}}{\frac{i-2}{2}} = 1 = \binom{\frac{i}{2}}{\frac{i}{2}}$, We get

$D_i = \binom{i-1}{0} + \{\binom{i-2}{1} + \binom{i-2}{0}\} + \{\binom{i-3}{2} + \binom{i-3}{1}\} + \cdots + +\{\binom{\frac{i}{2}+1}{\frac{i}{2}-1} + \binom{\frac{i}{2}+1}{\frac{i}{2}-2}\} + \binom{\frac{i-2}{2}}{\frac{i-2}{2}}$

$= D_{i-1} + D_{i-2}$

For Example:

$$D_6 = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3}$$

$$\binom{5}{0} + \{\binom{4}{1} + \binom{4}{0}\} + \{\binom{3}{2} + \binom{3}{1}\} + \binom{2}{2}$$

$$= \{\binom{5}{0} + \binom{4}{1} + \binom{3}{2}\} + \{\binom{4}{0} + \binom{3}{1} + \binom{2}{2}\}$$

$$= D_5 + D_4$$

Case 2: Similarly, when i is odd:

$$D_i = \binom{i}{0} + \binom{i-1}{1} + \binom{i-2}{2} + \cdots + \binom{\frac{i+1}{2}}{\frac{i-1}{2}}$$

$$= \{\binom{i-1}{0}\} + \{\binom{i-2}{1}\} + \binom{i-2}{0}\} + \{\binom{i-3}{2} + \binom{i-3}{1}\} + \cdots + \{\binom{\frac{i-1}{2}}{\frac{i-1}{2}} + \binom{\frac{i-1}{2}}{\frac{i-2}{2}}\}$$

$$= D_{i-1} + D_{i-2}$$

2. **Brualdi 5.7**

Use the binomial theorem to prove that

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Generalize to find the sum

$$\sum_{k=0}^{n} \binom{n}{k} r^k$$

for any real number r.

Answer:

By the binomial theorem we get

$$3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Similarly, for the second part we get

$$\sum_{k=0}^{n} \binom{n}{k} r^k = (1+r)^n$$

3. **Brualdi 5.10** Use combinatorial reasoning to prove identity (5.2).

   Answer:

   Identity 5.2 is:

   $$k\binom{n}{k} = n\binom{n-1}{k-1}$$

   We can choose a team of k players from n players in $\binom{n}{k}$ ways. After choosing the team there are k ways of choosing a team leader. Hence there are $k\binom{n}{k}$ ways of choosing a team and a leader of the team.

   On the other hand, We can choose the leader first from the n players in n ways. After choosing the leader, there are $\binom{n-1}{k-1}$ ways of choosing the rest of the team. Thus we have $n\binom{n-1}{k-1}$ ways of choosing a team and a leader of the team.

   This also proves the identity.

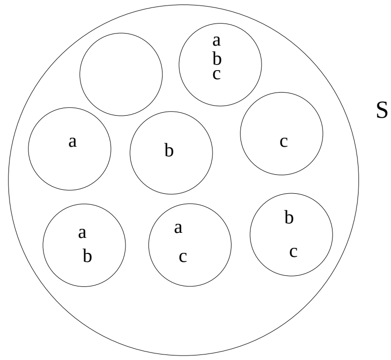
4. **Brualdi 5.11**

   Use Combinatorial reasoning to prove the identity

   $$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$$

   Answer:

   Let S be a set with three distinguished elements a, b and c. Then the type of sets with k elements in S are sets that do not contain a, b, or c or they contain only a, b, or c or they contain a, b and not c or a, c and not b or b, c and not a (see figure).



The sets of cardinality k that do not contain a, b or c are sets that are chosen from the $n-3$ elements that do not include a, b and c and there are $\binom{n-3}{k}$ of them.

Therefore $\binom{n}{k} - \binom{n-3}{k}$ is the number of ways we can choose a k-set that includes at least one of the distinguished elements, a, b and c.

Now consider sets that will always contain a. we can choose such a set by choosing $k-1$ elements from the $n-1$ elements that do not include a and then add a to get a set with k elements. There are $\binom{n-1}{k-1}$ such sets. Note that these are the sets that contain $\{a\}$ or $\{a, b\}$ or $\{a, c\}$ or $\{a, b, c\}$.

Now consider the sets that will not have a but will always have b. we can choose such a set by choosing $k-1$ elements from the $n-2$ elements that do not contain a and b and then add b. There are $\binom{n-2}{k-1}$ of such sets. Note that these are the sets that contain $\{b\}$ or $\{b, c\}$.

Similarly we will consider sets that always contain c but do not contain a or b and there are $\binom{n-3}{k-1}$ such sets.

Thus $\binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$ is also the number of ways we can choose a k-set that includes at least one of the distinguished elements, a, b and c.

This gives us the proof of the identity.

5. **Brualdi 5.18**

Evaluate the sum

$$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \cdots + (-1)^n \frac{1}{n+1}\binom{n}{n}$$

Answer:

$1-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\frac{1}{4}\binom{n}{3}+\cdots+(-1)^n\frac{1}{n+1}\binom{n}{n}$

$=\sum_{k=0}^{n}(-1)^k\frac{1}{k+1}\binom{n}{k}$

From the identity 5.2, $k\binom{n}{k}=n\binom{n-1}{k-1}$

We get $\frac{1}{n}\binom{n}{k}=\frac{1}{k}\binom{n-1}{k-1}$

Therefore $\sum_{k=0}^{n}(-1)^k\frac{1}{k+1}\binom{n}{k}=\sum_{k=0}^{n}(-1)^k\frac{1}{n+1}\binom{n+1}{k+1}$

$=\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}$

We will now apply identity 5.4, $\sum_{i=0}^{n}(-1)^k\binom{n}{k}=0$ after adjusting the above summation. Substitute $k+1=i$, then when $k=0, i=1$ and when $k=n, i=n+1$. Therefore, we get

$\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}$

$=\frac{1}{n+1}\sum_{i=1}^{n+1}(-1)^{i-1}\binom{n+1}{i}$

$=\frac{1}{n+1}(-1)\sum_{i=1}^{n+1}(-1)^{i}\binom{n+1}{i}$

$=\frac{1}{n+1}(-1)\{\sum_{i=0}^{n+1}(-1)^{i}\binom{n+1}{i}-1\}$

$=\frac{1}{n+1}(-1)\{0-1\}$

$=\frac{1}{n+1}$

Hence

$$1-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\frac{1}{4}\binom{n}{3}+\cdots+(-1)^n\frac{1}{n+1}\binom{n}{n}=\frac{1}{n+1}.$$

An alternating proof is obtained by using calculus. First consider the function

$$f(x)=x-\frac{1}{2}\binom{n}{1}x^2+\frac{1}{3}\binom{n}{2}x^3-\frac{1}{4}\binom{n}{3}x^4+\cdots+(-1)^n\frac{1}{n+1}\binom{n}{n}x^{n+1}\ .$$

Our objective is to evaluate $f(1)$. Now, it is easy to see that Noticing the fact that

$f(0) = 0$ by definition, we conclude that

$$\begin{aligned} f(x) &= \int_0^x f'(t)dt \\ &= \int_0^x (1-t)^n dt \\ &= -\frac{1}{n+1}(1-x)^{n+1} + \frac{1}{n+1} . \end{aligned}$$

Here we used the equality $f(0) = 0$ to determine the constant of integration. From this formula we conclude that $f(1) = \frac{1}{n+1}$.

6. **Brualdi 5.19**

   Sum the series $1^2 + 2^2 + 3^2 + \cdots + n^2$ by observing that

   $$m^2 = 2\binom{m}{2} + \binom{m}{1}$$

   and using the identity 5.14.

   Answer:

   Note that
   $$2\binom{m}{2} + \binom{m}{1} = 2\frac{m(m-1)}{2!} + m = m^2$$

   We can write $1^2 + 2^2 + 3^2 + \cdots + n^2 = 0^2 + 1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{m=0}^{n} m^2$.

   By applying the identity 5.14, $\sum_{m=0}^{n} \binom{m}{k} = \binom{n+1}{k+1}$, we get

   $$\begin{aligned} &\textstyle\sum_{m=0}^{n} m^2 = \sum_{m=0}^{n} 2\binom{m}{2} + \sum_{m=0}^{n} \binom{m}{1} \\ &= 2\binom{n+1}{3} + \binom{n+1}{2}. \end{aligned}$$

7. **Brualdi 5.20**

   Find integers a, b and c such that

   $$m^3 = a\binom{m}{3} + b\binom{m}{2} + c\binom{m}{1}$$

   for all m. Then sum the series $1^3 + 2^3 + 3^3 + \cdots + n^3$.

   Answer:

$$a\binom{m}{3} + b\binom{m}{2} + c\binom{m}{1} - m^3 = 0$$

$$a\frac{m!}{3!(m-3)!} + b\frac{m!}{2!(m-2)!} + c\frac{m!}{1!(m-1)!} - m^3 = 0$$

$$a\frac{m(m-1)(m-2)}{6} + b\frac{m(m-1)}{2} + c\frac{m}{1} - m^3 = 0$$

$$\frac{m^3(a-6)+m^2(-3a+3b)+m(2a-3b+6c)}{6} = 0$$

Therefore we get $a = 6, b = 6, c = 1$. Also

$$\sum_{m=1}^{n} m^3 = \sum_{m=1}^{n} \left[6\binom{m}{3} + 6\binom{m}{2} + \binom{m}{1}\right]$$

The identity 5.14, $\sum_{m=0}^{n}\binom{m}{k} = \binom{n+1}{k+1}$ implies $\sum_{m=1}^{n}\binom{m}{k} = \binom{n+1}{k+1}$, since $\binom{0}{k} = 0$. Therefore we get

$$\sum_{m=1}^{n} m^3 = \sum_{m=1}^{n} 6\binom{m}{3} + \sum_{m=1}^{n} 6\binom{m}{2} + \sum_{m=1}^{n}\binom{m}{1}$$

$$= 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2}$$

8. **Brualdi 5.23** Every day a student walks from her home to school, which is located 10 blocks east and 14 blocks north from home. She always takes a shortest walk of 24 blocks.

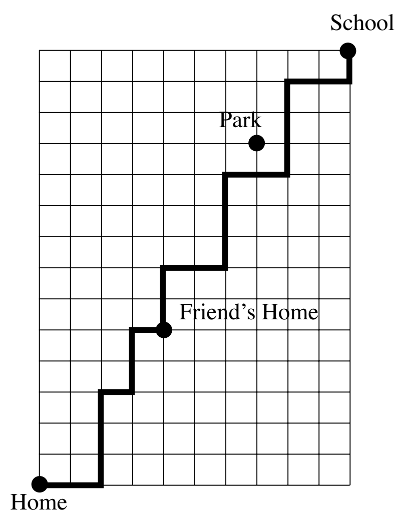


Figure 1: A shortest path from the student's home to her school, stopping by at her friend's home but avoiding to go through the park.

a) How many different walks are possible?

Answer:

Let walking a block in the east direction be called a east step and walking a block in the north direction be called a north step. Different walks corresponds to different combinations of east and north steps. (see figure).

Hence the number of ways student can walk a total of 24 blocks using 10 east steps and 14 north steps is $\binom{24}{10} = \frac{24!}{10!14!}$.

b) Suppose that 4 blocks east and 5 blocks north of her home lives her best friend, whom she meets each day on her way to school. Now how many different walks are possible?

Answer:

The first 9 blocks to her friend's place she can walk in $\frac{9!}{4!5!}$ ways. Now there are 15 blocks left and 6 east steps and 9 north steps and so she can walk these 15 steps in $\frac{15!}{6!9!}$ ways.

Hence the total number of walks is

$$\binom{9}{4}\binom{15}{6} = \frac{9!}{4!5!} \cdot \frac{15!}{6!9!} = \frac{15!}{4!5!6!} .$$

c) Suppose in addition, that 3 blocks east and 6 blocks north of her friend's house there is a park where the two girls stop each day to rest and play. Now how many different walks are there?

Answer:

The first 9 blocks to her friend's place the student can walk in $\frac{9!}{4!5!}$ ways. The Second 9 blocks to the park, the two girls can walk in $\frac{9!}{3!6!}$ ways. The remaining 6 blocks can be walked in $\frac{6!}{3!3!}$ ways. Hence there are

$$\binom{9}{4}\binom{9}{3}\binom{6}{3} = \frac{9!}{4!5!}\frac{9!}{3!6!}\frac{6!}{3!3!} = \frac{9!^2}{4!5!3!^3}$$

such walks.

d) Stopping at a park to rest and play, the students often get to school late. To avoid temptation of the park, our two students decide never to pass the intersection where the park is. Now how many different walks are there?

Answer:

This is complement of part c. Therefore there are

$$\binom{9}{4}\binom{15}{6} - \binom{9}{4}\binom{9}{3}\binom{6}{3} = \frac{15!}{4!5!6!} - \frac{9!^2}{4!5!3!^3}$$

such walks.

9. **Brualdi 5.25**

   Use a combinatorial argument, to prove the Vandermonde convolution for the binomial coefficients: for all positive integers m_1, m_2 and n,

$$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n}$$

   Deduce the identity 5.11 as a special case.

   Answer:

   Let m_1 be the number of items of type A (say candies) and let m_2 be the number of items of type B (say ice creams). $\binom{m_1+m_2}{n}$ is the number of ways you can choose n items from $m_1 + m_2$ items.

   We can choose k items of type A in $\binom{m_1}{k}$ ways and $n-k$ items of type B in $\binom{m_2}{n-k}$ ways, where $k = 0, 1, 2, \ldots, n$.

   Therefore

$$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n}.$$

   It we let $m_1 = m_2 = n$, we get

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

   But we know that $\binom{n}{k} = \binom{n}{n-k}$ (Problem 3.15). Therefore we get the identity 5.11:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

10. **Brualdi 5.26** Find and prove a formula for

$$\sum_{\substack{r,s,t \geq 0 \\ r+s+t=n}} \binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t}$$

Answer:

Claim:

$$\sum_{\substack{r,s,t\geq 0\\ r+s+t=n}} \binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t} = \binom{m_1+m_2+m_3}{n}.$$

Proof:

By repeated application of Vandermonde convolution formula (problem 5.25), we get

$$\tbinom{m_1+m_2+m_3}{n} = \sum_{k=0}^{n} \tbinom{m_1}{k}\tbinom{m_2+m_3}{n-k}$$

$$= \sum_{k=0}^{n} \tbinom{m_1}{k} \sum_{j=0}^{n-k} \tbinom{m_2}{j}\tbinom{m_3}{n-k-j}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \tbinom{m_1}{k}\tbinom{m_2}{j}\tbinom{m_3}{n-k-j}$$

$$= \sum_{\substack{r,s,t\geq 0\\ r+s+t=n}} \tbinom{m_1}{r}\tbinom{m_2}{s}\tbinom{m_3}{t}.$$

A combinatorial proof is also easy. We have three different sets, consisting of m_1 boys, m_2 girls, and m_3 dogs, say. The quantity in question is the number of ways to form a subset consisting of n mammals. A subset of n mammals has r boys, s girls, and t dogs, where $r+s+t=n$. If we exhausts all possibilities of braking down a mammal set into its three components, you obtain the formula.