MAT 145: Homework Solutions #4

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1. **Brualdi 2.9**

   In a room there are 10 people, none of whom are older than 60 (ages are given in whole numbers only) but each of whom is at least 1 year old. Prove that one can always find two groups of people (with no common person) the sum of whose ages is the same. Can 10 be replaced by a smaller number?

   Answer:

   The number of ways we can select a non-empty group of people from 10 people is $2^{10} - 1 = 1024 - 1 = 1023$. The maximum age sum possible for a group of people is $10 \times 60 = 600$. Since the number of groups is greater than the number of sums possible, by pigeon hole principle at least two non-empty groups have the same sum. If these groups have common persons, then by removing them from each group, we obtain two disjoint subset of the 10 people with the same total age.

   The pigeonhole principle, in its exact same way used as above, does not apply for 9 people because $2^9 = 512$ is not greater than $9 \times 60 = 540$. But this proves nothing. Actually, even the total number of people is reduced to 9, you *can* always find two disjoint subsets with the same total age! This fact requires more elaborate argument to prove.

   Suppose there were a set of 9 people where no two subsets have the same total age. Then all $2^9 = 512$ subsets have distinct values of total age. Since no two people have the same age (otherwise you do have two subsets with the same total age), actually the maximum total age of these 9 people cannot exceed $60+59+58+57+56+55+54+53+52 = 504$. Since $512 - 1 = 511 > 504$, the 512 subsets cannot take distinct values from 0 to 504. Contradiction!

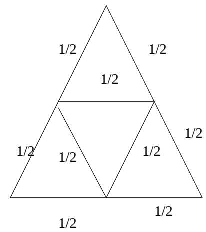
   How about 8 people? We compare $2^8 = 256$ and $60+59+58+57+56+55+54+53 = 452$. Even our new argument does not apply here. What can we say, then? *The problem is stupid!* Right?

2. **Brualdi 2.19**

   a) Prove that of any points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most 1/2.

Answer:

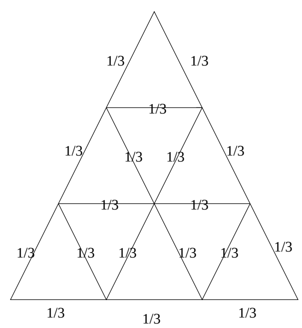
Divide the triangle into 4 small triangle of side length 1/2 as shown in the figure. Then any two points chosen within one of the smaller triangles are at most 1/2 units apart. By Pigeonhole principle, if we put 5 points in 4 triangles, two points are in the same triangle. Hence we are done.



b) Prove that any ten points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most 1/3.

Answer:

Divide the triangle into 9 smaller triangles of side length 1/3 as shown in the figure. By the pigeonhole principle at least 2 of the 10 points are in the same triangle.



c) Determine an integer m_n such that if m_n points are chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $1/n$.

Answer:

$m_n = n^2 + 1$. Divide the triangle into n^2 congruent equilateral triangles each of side $1/n$.

3. **Brualdi 3.4**

How many positive divisors do each of the following numbers have

a) $3^4 \times 5^2 \times 7^6 \times 11$

Answer:

The divisors of $3^4 \times 5^2 \times 7^6 \times 11$ are of the form $3^i \times 5^j \times 7^k \times 11^l$, where $i \in \{0,1,2,3,4\}, j \in \{0,1,2\}, k \in \{0,1,2,3,4,5,6\}, l \in \{0,1\}$.

Hence there are $5 \cdot 3 \cdot 7 \cdot 2 = 210$ divisors of the given number.

b) 620

Answer:

Since $620 = 2^2 \cdot 5^1 \cdot 31$, arguing as in part (a) of the problem, we get that there are $3 \times 2 \times 2 = 12$ divisors of 620.

c) 10^{10}

Answer:

Now $10^{10} = 2^{10}5^{10}$. This implies, like in part (a), that 10^{10} has $11 \cdot 11 = 121$ divisors.

4. **Brualdi 3.6**

How many integers greater than 5400 have both of the following properties?

a) The digits are distinct.

b) The digits 2 and 7 do not occur.

Answer:

Let $a_1a_2...a_n$ be an integer with $n > 4$ distinct integers (digits). Note such a number is automatically bigger than 5400. a_1 cannot be 0,2 or 7 and hence there are 7 choices for a_1. The other digits cannot be 2 or 7 and there are 8 choices for each of them. Hence the number of integers with $n > 4$ distinct digits is $7P(7, n-1)$. Since $n > 4$, we can conclude that $n = 5, 6, 7, 8$.

Similarly, the number of integers with $n = 4$ distinct digits and such that the digits 2 and 7 do not occur is $7P(7,3)$. If $a_1a_2a_3a_4$ with $a_1 \neq 0$ is such an integer and is less than 5400 then either a_1 is 1,3 or 4 and the other three digits cannot be 2 or 7 and there are $3P(7,3)$ such numbers or $a_1 = 5$ and $a_2 = 0, 1$ or 3 and the rest of the numbers cannot be 2 or 7. Hence there are $3P(6,2)$ such numbers. Hence the number of integers with $n = 4$ distinct digits and such that the digits 2 and 7 do not occur is $7P(7,3) - 3P(7,3) - 3P(6,2)$

Hence there are

$$\{\sum_5^8 7P(7, n-1)\} + \{7P(7,3) - 3P(7,3) - 3P(6,2)\}$$

5. **Brualdi 3.14**

At a party there are 15 men and 20 women.

(a) How many ways are there to form 15 couples consisting of one man and one woman?

Answer:

There are $\binom{20}{15}$ ways of choosing women for the 15 couples and $\binom{15}{15}$ ways of choosing the men and 15! ways of coupling the 15 chosen women with the 15 men. Hence there are a total of

$$\binom{20}{15} \cdot \binom{15}{15} \cdot 15!$$

$$= \frac{20!}{15! \cdot 5!} \cdot 15!$$

$$= \frac{20!}{5!}.$$

(b) How many ways are there to form 10 couples consisting of one man and one woman?

Answer:

Using a similar argument as above we have that the number of ways of forming 10 couples consisting of one man and one woman is $\binom{20}{10} \cdot \binom{15}{10} \cdot 10!$

6. **Brualdi 3.37**

Prove that the number of ways to distribute n different objects among k children equals k^n.

Answer:

Any object can be distributed in k ways since there are k children. Hence the number of ways to distribute n different objects among k children equals k^n.

7. **Brualdi 5.12**

Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^m \binom{2m}{m} & \text{if } n = 2m. \end{cases}$$

Answer:

Probably it is easy for you to see why the value is 0 when n is odd. But to illustrate what's going on behind the scene, let me give you a proof that solves both odd and even cases simultaneously.

Consider the binomial expansion

$$(1 - x^2)^n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} x^{2r} \ . \tag{1}$$

We first notice that this expansion does not contain any odd power terms. Now, using the factorization $(1-x^2) = (1-x)(1+x)$, we have

$$\begin{aligned}(1-x^2)^n = (1-x)^n(1+x)^n &= \left(\sum_{p=0}^{n}(-1)^p\binom{n}{p}x^p\right)\left(\sum_{q=0}^{n}\binom{n}{q}x^q\right) \\ &= \sum_{s=0}^{2n}\left(\sum_{p=0}^{s}(-1)^p\binom{n}{p}\binom{n}{s-p}\right)\cdot x^s \ ,\end{aligned} \tag{2}$$

where $s = p+q$. Since the two expansion formulas should agree, comparing the coefficients of Eqn.(1) and Eqn.(2) term by term, we obtain for every $s \geq 0$

$$\sum_{p=0}^{s}(-1)^p\binom{n}{p}\binom{n}{s-p} = \begin{cases} 0 & \text{if } s \text{ is odd} \\ (-1)^r\binom{n}{r} & \text{if } s = 2r\ . \end{cases} \tag{3}$$

The desired formula in the book is established by specializing Eqn.(3) for $s = n$, noticing $\binom{n}{n-p} = \binom{n}{p}$.

8. **Brualdi 5.13**

Find one binomial coefficient equal to the following expression

$$\binom{n}{k} + 3\binom{n}{k-1} + 3\binom{n}{k-2} + \binom{n}{k-3}$$

Answer:

The required binomial coefficient is $\binom{n+3}{k}$ because by repeated application Pascal's formula we get

$$\begin{aligned}\tbinom{n+3}{k} &= \tbinom{n+2}{k} + \tbinom{n+2}{k-1} \\ &= \left\{\tbinom{n+1}{k} + \tbinom{n+1}{k-1}\right\} + \left\{\tbinom{n+1}{k-1} + \tbinom{n+1}{k-2}\right\} \\ &= \tbinom{n+1}{k} + 2\tbinom{n+1}{k-1} + \tbinom{n+1}{k-2} \\ &= \left\{\tbinom{n}{k} + \tbinom{n}{k-1}\right\} + 2\left\{\tbinom{n}{k-1} + \tbinom{n}{k-2}\right\} + \left\{\tbinom{n}{k-2} + \tbinom{n}{k-3}\right\} \\ &= \tbinom{n}{k} + 3\tbinom{n}{k-1} + 3\tbinom{n}{k-2} + \tbinom{n}{k-3}\ .\end{aligned}$$

There is also a simple combinatorial proof. Consider a set consisting of n boys and 3 girls. We wish to count the number of ways to form a group of k people. Of course it is $\binom{n+3}{k}$. Every group is classified into four categories: groups with no girls, one girl, two girls and three girls. The desired formula follows from this classification.

9. **Brualdi 5.41**

   Use Newton's binomial theorem to approximate $\sqrt{30}$.

   Answer:

   By the binomial theorem we have when $|z| < 1$,

$$\sqrt{1+z} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k2^{2k-1}} \binom{2k-2}{k-1} z^k$$

   So we get

$$\sqrt{30} = \sqrt{25+5} = \sqrt{25(1+\tfrac{1}{5})} = 5(1+\tfrac{1}{5})^{\frac{1}{2}}$$

$$= 5(1 + \tfrac{1}{2}(0.2) - \tfrac{1}{8}(0.2)^2 + \tfrac{1}{16}(0.2)^3 - \cdots)$$

$$= 5(1 + 0.1 - 0.005 + 0.0005 - \cdots)$$

$$= 5 + 0.5 - 0.025 + 0.0025 - \cdots$$

$$= 5.477...$$

10. **Brualdi 5.42**

   Use Newton's binomial theorem to approximate $10^{\frac{1}{3}}$.

   Answer:

$$10^{\frac{1}{3}} = (8+2)^{\frac{1}{3}} = (8(1+\tfrac{1}{4}))^{\frac{1}{3}} = 2(1+\tfrac{1}{4})^{\frac{1}{3}}$$

$$= 2\left(\textstyle\sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} (\frac{1}{4})^k\right)$$

$$= 2\left(1 + \tfrac{1}{3}\tfrac{1}{4} + \tfrac{1/3(1/3-1)}{2}\tfrac{1}{16} + \tfrac{1/3(1/3-1)(1/3-2)}{6}\tfrac{1}{64} - \cdots\right)$$

$$= 2 + \tfrac{1}{6} - \tfrac{1}{8 \cdot 9} + \tfrac{5}{32 \cdot 81} - \cdots$$

$$= 2.154...$$