

MAT 145: Homework Solutions #6

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May 8, 2003

1. Brualdi 6.2

Find the number of integers between 1 and 10,000 inclusive which are not divisible by 4,6,7, or 10.

Answer:

Let A_1 be the set of integers between 1 and 10,000 that are divisible by 4, A_2 be the set of integers between 1 and 10,000 that are divisible by 6, A_3 be the set of integers between 1 and 10,000 that are divisible by 7 and let A_4 be the set of integers between 1 and 10,000 that are divisible by 10. Then by the inclusion-exclusion principle the number of integers between 1 and 10,000 inclusive which are not divisible by 4,6,7, or 10 is

$$\begin{aligned} & |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| \\ &= 10,000 - (|A_1| + |A_2| + |A_3| + |A_4|) \\ & \quad + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\ & \quad - (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) \\ & \quad - |A_1 \cap A_2 \cap A_3 \cap A_4| \\ &= 10,000 - (2500 + 1666 + 1428 + 1000) + (833 + 357 + 500 + 238 + 333 + 142) \\ & \quad - (119 + 166 + 71 + 47) + 23 \\ &= 5429. \end{aligned}$$

because:

$$\begin{aligned}
|A_1| &= \lfloor \frac{10,000}{4} \rfloor = 2500 \\
|A_2| &= \lfloor \frac{10,000}{6} \rfloor = 1666 \\
|A_3| &= \lfloor \frac{10,000}{7} \rfloor = 1428 \\
|A_4| &= \lfloor \frac{10,000}{10} \rfloor = 1000 \\
|A_1 \cap A_2| &= \lfloor \frac{10,000}{\text{lcm}\{4,6\}} \rfloor = \lfloor \frac{10,000}{12} \rfloor = 833 \\
|A_1 \cap A_3| &= \lfloor \frac{10,000}{\text{lcm}\{4,7\}} \rfloor = \lfloor \frac{10,000}{28} \rfloor = 357 \\
|A_1 \cap A_4| &= \lfloor \frac{10,000}{\text{lcm}\{4,10\}} \rfloor = \lfloor \frac{10,000}{20} \rfloor = 500 \\
|A_2 \cap A_3| &= \lfloor \frac{10,000}{\text{lcm}\{6,7\}} \rfloor = \lfloor \frac{10,000}{42} \rfloor = 238 \\
|A_2 \cap A_4| &= \lfloor \frac{10,000}{\text{lcm}\{6,10\}} \rfloor = \lfloor \frac{10,000}{30} \rfloor = 333 \\
|A_3 \cap A_4| &= \lfloor \frac{10,000}{\text{lcm}\{7,10\}} \rfloor = \lfloor \frac{10,000}{70} \rfloor = 142 \\
|A_1 \cap A_2 \cap A_3| &= \lfloor \frac{10,000}{\text{lcm}\{4,6,7\}} \rfloor = \lfloor \frac{10,000}{84} \rfloor = 119 \\
|A_1 \cap A_2 \cap A_4| &= \lfloor \frac{10,000}{\text{lcm}\{4,6,10\}} \rfloor = \lfloor \frac{10,000}{60} \rfloor = 166 \\
|A_1 \cap A_3 \cap A_4| &= \lfloor \frac{10,000}{\text{lcm}\{4,7,10\}} \rfloor = \lfloor \frac{10,000}{140} \rfloor = 71 \\
|A_2 \cap A_3 \cap A_4| &= \lfloor \frac{10,000}{\text{lcm}\{6,7,10\}} \rfloor = \lfloor \frac{10,000}{210} \rfloor = 47 \\
|A_1 \cap A_2 \cap A_3 \cap A_4| &= \lfloor \frac{10,000}{\text{lcm}\{4,6,7,10\}} \rfloor = \lfloor \frac{10,000}{420} \rfloor = 23
\end{aligned}$$

2. Brualdi 6.3

Find the number of integers between 1 and 10,000 which are neither perfect squares nor perfect cubes.

Answer:

Let $S = \{1, 2, \dots, 10000\}$ be the set of all integers between 1 and 10,000. Then $|S| = 10000$. Let A_1 be the set of all perfect squares in S . Then $|A_1| = \sqrt{10000} = 100$ since all integers less than 100 and their perfect squares are in S . Let A_2 be the set of all perfect cubes in S . Then $|A_2| = 9261^{1/3} = 21$ since 9261 is the largest number that is a perfect cube in S . Now $A_1 \cap A_2$ are integers in S that are both perfect squares

and perfect cubes. Therefore if n is in $A_1 \cap A_2$, and the prime factorization of n is $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ then each exponent e_i is divisible by 6. That is $A_1 \cap A_2$ are integers in S that are 6th powers of an integer. The largest integer which is 6th power in S is 4096. so $|A_1 \cap A_2| = 4096^{1/6} = 4$.

Therefore by the inclusion-exclusion principle we have the number of integers between 1 and 10,000 which are neither perfect squares nor perfect cubes is

$$|S| - |A_1| - |A_2| + |A_1 \cap A_2| = 10,000 - 100 - 21 + 4 = 9883$$

3. Brualdi 6.7

Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 14$ in non-negative integers x_1, x_2, x_3 and x_4 not exceeding 8.

Answer:

Let S be the set of all non-negative integral solutions of the equation $x_1 + x_2 + x_3 + x_4 = 14$. Then, the number of non-negative integral solutions of the given equation is, $|S| = \binom{14+4-1}{14} = \binom{17}{14} = 680$.

Let the set A_1 consist of solutions in S for which $x_1 \geq 9$. We make a change of variable, $y_1 = x_1 - 9$ to get $|A_1|$ which is the same as the number of non-negative solutions of the equation $y_1 + x_2 + x_3 + x_4 = 5$. Therefore

$$|A_1| = \binom{5+4-1}{5} = \binom{8}{5} = 56.$$

Similarly, if we let A_2 be the set of solutions in S for which $x_2 \geq 9$, A_3 be the set of solutions in S for which $x_3 \geq 9$ and A_4 be the set of solutions in S for which $x_4 \geq 9$, we get

$$|A_2| = |A_3| = |A_4| = 56.$$

The set $A_1 \cap A_2$ consists of solutions in S which have $x_1 \geq 9$ and $x_2 \geq 9$. Let $y_1 = x_1 - 9, y_2 = x_2 - 9$. Then, $|A_1 \cap A_2|$ is the same as the non-negative integral solutions of the equation

$$y_1 + y_2 + x_3 + x_4 = -4$$

Thus, $|A_1 \cap A_2| = 0$. Similarly, we can easily verify that

$$|A_1 \cap A_3| = 0$$

$$|A_1 \cap A_4| = 0$$

$$|A_2 \cap A_3| = 0$$

$$|A_2 \cap A_4| = 0$$

$$|A_3 \cap A_4| = 0$$

$$|A_1 \cap A_2 \cap A_3| = 0$$

$$|A_1 \cap A_2 \cap A_4| = 0$$

$$|A_1 \cap A_3 \cap A_4| = 0$$

$$|A_2 \cap A_3 \cap A_4| = 0$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$$

By inclusion-exclusion principle we get that the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 14$ in non-negative integers x_1, x_2, x_3 and x_4 not exceeding 8 is

$$\begin{aligned} & |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| \\ &= |S| - (|A_1| + |A_2| + |A_3| + |A_4|) \\ &+ (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\ &- (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) + |A_1 \cap A_2 \cap A_3 \cap A_4| \\ &= 456. \end{aligned}$$

(It is quite obvious that the sets A_1, \dots, A_4 are disjoint because if two numbers are greater than 8, then the sum of them together with other non-negative integers exceeds 14. Thus $680 - 4 \times 56 = 456$.)

4. Brualdi 6.9

Determine the number of integral solutions of the equation $x_1 + x_2 + x_3 + x_4 = 20$, which satisfy

$$1 \leq x_1 \leq 6, 0 \leq x_2 \leq 7, 4 \leq x_3 \leq 8, 2 \leq x_4 \leq 6.$$

Answer:

Let $y_1 = x_1 - 1, y_2 = x_2, y_3 = x_3 - 4, y_4 = x_4 - 2$. Then the given equation becomes $y_1 + y_2 + y_3 + y_4 = 13$ with

$$0 \leq y_1 \leq 5, 0 \leq y_2 \leq 7, 0 \leq y_3 \leq 4, 0 \leq y_4 \leq 4.$$

Let S be the set of all non-negative integral solutions of the equation $y_1 + y_2 + y_3 + y_4 = 13$. Then, $|S| = \binom{13+4-1}{13} = \binom{16}{13} = 560$.

Let the set A_1 consist of solutions in S for which $y_1 \geq 6$. We make a change of variable, $z_1 = y_1 - 6$ to get $|A_1|$ which is the same as the number of non-negative solutions of the equation $z_1 + y_2 + y_3 + y_4 = 7$. Therefore

$$|A_1| = \binom{7+4-1}{7} = \binom{10}{7} = 120.$$

Let the set A_2 consist of solutions in S for which $y_2 \geq 8$. We make a change of variable, $z_2 = y_2 - 8$ to get $|A_2|$ which is the same as the number of non-negative solutions of the equation $y_1 + z_2 + y_3 + y_4 = 13 - 8 = 5$. Therefore

$$|A_2| = \binom{5+4-1}{5} = \binom{8}{5} = 56.$$

Let the set A_3 consist of solutions in S for which $y_3 \geq 5$. We make a change of variable, $z_3 = y_3 - 5$ to get $|A_3|$ which is the same as the number of non-negative solutions of the equation $y_1 + y_2 + z_3 + y_4 = 13 - 5 = 8$. Therefore

$$|A_3| = \binom{8+4-1}{8} = \binom{11}{8} = 165.$$

Let the set A_4 consist of solutions in S for which $y_4 \geq 5$. We make a change of variable, $z_4 = y_4 - 5$ to get $|A_4|$ which is the same as the number of non-negative solutions of the equation $y_1 + y_2 + y_3 + z_4 = 13 - 5 = 8$. Therefore

$$|A_4| = \binom{8+4-1}{8} = \binom{11}{8} = 165.$$

The set $A_1 \cap A_2$ consists of solutions in S which have $y_1 \geq 6$ and $y_2 \geq 8$. Let $z_1 = y_1 - 6, z_2 = y_2 - 8$. Then, $|A_1 \cap A_2|$ is the same as the non-negative integral solutions of the equation

$$z_1 + z_2 + y_3 + y_4 = 13 - 6 - 8 = -1$$

Thus, $|A_1 \cap A_2| = 0$.

The set $A_1 \cap A_3$ consists of solutions in S which have $y_1 \geq 6$ and $y_3 \geq 5$. Let $z_1 = y_1 - 6, z_3 = y_3 - 5$. Then, $|A_1 \cap A_3|$ is the same as the non-negative integral solutions of the equation

$$z_1 + y_2 + z_3 + y_4 = 13 - 6 - 5 = 2$$

Thus, $|A_1 \cap A_3| = \binom{2+4-1}{2} = \binom{5}{2} = 10$.

Similarly, it can be easily verified that

$$|A_1 \cap A_4| = 10$$

$$|A_2 \cap A_3| = 1$$

$$|A_2 \cap A_4| = 1$$

$$|A_3 \cap A_4| = 20$$

$$|A_1 \cap A_2 \cap A_3| = 0$$

$$|A_1 \cap A_2 \cap A_4| = 0$$

$$|A_1 \cap A_3 \cap A_4| = 0$$

$$|A_2 \cap A_3 \cap A_4| = 0$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$$

By inclusion-exclusion principle we get that the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 20$ which satisfy

$$1 \leq x_1 \leq 6, 0 \leq x_2 \leq 7, 4 \leq x_3 \leq 8, 2 \leq x_4 \leq 6.$$

is

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}|$$

$$= |S| - (|A_1| + |A_2| + |A_3| + |A_4|)$$

$$+ (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|)$$

$$- (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) + |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$$= 96.$$

5. **Brualdi 6.11**

Determine the number of permutations of $\{1, 2, \dots, 8\}$ in which no even integer is in its natural position.

Answer:

Let S be the set of all permutations of $\{1, 2, \dots, 8\}$. Then, $|S| = 8!$. Let A_1 be the set of all permutations of $\{1, 2, \dots, 8\}$ such that 2 is in its natural position. Thus $|A_1| = 7!$. Similarly, let A_2 be the set of all permutations of $\{1, 2, \dots, 8\}$ such that 4 is in its natural position, A_3 be the set of all permutations of $\{1, 2, \dots, 8\}$ such that 6 is in its natural position. and let A_4 be the set of all permutations of $\{1, 2, \dots, 8\}$ such that 8 is in its natural position, then we get

$$|A_2| = |A_3| = |A_4| = 7!$$

$A_1 \cap A_2$ is the set of all permutations of $\{1, 2, \dots, 8\}$ such that 2 and 4 is in their natural position, therefore $|A_1 \cap A_2| = 6!$. Similarly, it can be easily verified that

$$|A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4| = 6!$$

$$|A_1 \cap A_2 \cap A_3| = |A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = 5!$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = 4!$$

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}|$$

$$= |S| - (|A_1| + |A_2| + |A_3| + |A_4|)$$

$$+ (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|)$$

$$- (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) + |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$$= 8! - 4 \cdot 7! + 6 \cdot 6! - 4 \cdot 5! + 4!$$

$$= 24024.$$

6. **Brualdi 6.12**

Determine the number of permutations of $\{1, 2, \dots, 8\}$ in which exactly four integers are in their natural position.

Answer:

The number of ways to choose 4 integers such that they are in their natural position is $\binom{8}{4}$. The other 4 integers form derangements. So the number of permutations of $\{1, 2, \dots, 8\}$ in which exactly four integers are in their natural positions is

$$\binom{8}{4} \cdot D_4.$$

7. Brualdi 6.14

Determine a general formula for the number of permutations of the set $\{1, 2, \dots, n\}$ in which exactly k integers are in their natural positions.

Answer:

The number of ways to choose k integers such that they are in their natural position is $\binom{n}{k}$. The other $n - k$ integers form derangements. So the number of permutations of $\{1, 2, \dots, n\}$ in which exactly k integers are in their natural position is

$$\binom{n}{k} \cdot D_{n-k}.$$

8. Brualdi 6.16

Use combinatorial reasoning to derive the identity

$$n! = \binom{n}{0}D_n + \binom{n}{1}D_{n-1} + \cdots + \binom{n}{n-1}D_1 + \binom{n}{n}D_0$$

Define $D_0 = 1$.

Answer:

We can partition the permutations according to the number of integers in their natural position. Since the number of permutations in which exactly k integers are in their natural positions is $\binom{n}{k} \cdot D_{n-k}$, we get that the total number of permutations

$$n! = \sum_{k=0}^n \binom{n}{k} \cdot D_{n-k}$$

9. **Brualdi 6.20** Starting from the formula $D_n = nD_{n-1} + (-1)^n$, ($n = 2, 3, 4, \dots$), give a proof of Theorem 6.3.1.

Answer:

We need to prove for $n \geq 1$

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right)$$

Proof is by induction:

$$D_2 = 2D_1 + (-1)^2 = 0 + 1 = 1$$

$$2!(1 - 1/1! + 1/2!) = 1$$

$$\text{Therefore } D_2 = 2!(1 - 1/1! + 1/2!)$$

Assume that the Theorem is true for $n = k$. Then we get

$$\begin{aligned} D_{k+1} &= (k+1)D_k + (-1)^{k+1} \\ &= (k+1)\{k!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^k \frac{1}{k!})\} + (-1)^{k+1} \text{ (by induction hypothesis)} \\ &= (k+1)!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{(k+1)} \frac{1}{(k+1)!}) \end{aligned}$$

10. Brualdi 6.21

Prove that D_n is an even number iff n is an odd number.

Answer:

(\Rightarrow) If D_n is an even number, then $D_n - (1)^n$ is odd. Since $D_n = nD_{n-1} + (-1)^n$, $n = 2, 3, \dots$, (see Problem 6.20), we get nD_{n-1} is odd and hence n is odd.

(\Leftarrow) Proof by contradiction: Let n be an odd number and let D_n be an odd number. Then, since $D_n - (1)^n$ is even and since $D_n = nD_{n-1} + (-1)^n$, $n = 2, 3, \dots$, we get nD_{n-1} is even. But since n is odd, we get that D_{n-1} is even. Therefore by (\Rightarrow), $n - 1$ is odd. Therefore n is even. This is a contradiction. Therefore D_n is an even number whenever n is an odd number.