# MAT 145: Homework Solutions #7

Prepared by Maya Ahmed

May 16, 2003

1. **Brualdi 7.1**

   Let $f_0, f_1, \ldots, f_n, \ldots$ denote the Fibonacci sequence. By evaluating each of the following expressions for small values of $n$, conjecture a general formula and then prove it, using mathematical induction and Fibonacci recurrence.

   (a)
   $$f_1 + f_3 + \cdots + f_{2n-1}$$

   **Answer:**

   observe that
   $$f_1 = 1 = f_2$$
   $$f_1 + f_3 = 1 + 2 = 3 = f_4$$
   $$f_1 + f_3 + f_5 = 1 + 2 + 5 = 8 = f_6$$

   So we make the conjecture that
   $$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}.$$

   We will prove the conjecture by using induction. We already checked base cases. Assume the conjecture is true for $n = k$. We need to show that the conjecture is true for $n = k + 1$.

   $$f_1 + f_3 + \cdots + f_{2(k+1)-1}$$
   $$= \{f_1 + f_3 + \cdots + f_{2k-1}\} + f_{2k+1}$$
   $$= \{f_{2k}\} + f_{2k+1} \text{ (by induction hypothesis)}$$
   $$= f_{2k+2} \text{ (by definition).}$$

   This proves the conjecture.

   (b)
   $$f_0 + f_2 + \cdots + f_{2n}$$

   **Answer:**

observe that

$$f_0 = 0 = f_1 - 1$$

$$f_0 + f_2 = 0 + 1 = 1 = f_3 - 1$$

$$f_0 + f_2 + f_4 = 0 + 1 + 3 = 4 = f_5 - 1$$

So we make the conjecture that

$$f_0 + f_2 + \cdots + f_{2n} = f_{2n+1} - 1.$$

We will prove the conjecture by using induction. We already checked base cases. Assume the conjecture is true for $n = k$. We need to show that the conjecture is true for $n = k + 1$.

$$f_0 + f_2 + \cdots + f_{2(k+1)}$$

$$= \{f_0 + f_2 + \cdots + f_{2k}\} + f_{2k+2}$$

$$= \{f_{2k+1} - 1\} + f_{2k+2}$$

$$= f_{2k+3} - 1$$

This proves the conjecture.

(c)

$$f_0 - f_1 + f_2 - \cdots + (-1)^n f_n$$

**Answer:**

observe that

$$f_0 = 0 = f_1 - 1 - f_0$$

$$f_0 - f_1 = 0 - 1 = -1 = f_1 - 1 - f_2$$

$$f_0 - f_1 + f_2 = 0 - 1 + 1 = 0 = f_3 - 1 - f_2$$

$$f_0 - f_1 + f_2 - f_3 = 0 - 1 + 1 - 2 = -2 = f_3 - 1 - f_4$$

So we make the conjecture that

$$f_0 - f_1 + f_2 - \cdots + (-1)^n f_n = \begin{cases} (f_n - 1) - f_{n+1} \text{ if } n \text{ is odd} \\ (f_{n+1} - 1) - f_n \text{ if } n \text{ is even} \end{cases}$$

We will prove the conjecture by using induction. We already checked base cases. Assume the conjecture is true for $n = k$. We need to show that the conjecture is true for $n = k + 1$.

Let $k$ be even. Then $k + 1$ is odd.

$$f_0 - f_1 + f_2 - \cdots + (-1)^{k+1} f_{k+1}$$

$$= \{f_0 - f_1 + f_2 - \cdots + f_k\} - f_{k+1}$$

$$= \{(f_{k+1} - 1) - f_k\} - f_{k+1} \text{ (by induction hypothesis)}$$

$$= (f_{k+1} - 1) - f_{k+2}$$

Let $k$ be odd. Then $k + 1$ is even.

$$f_0 - f_1 + f_2 - \cdots + (-1)^{k+1} f_{k+1}$$

$$= \{f_0 - f_1 + f_2 - \cdots - f_k\} + f_{k+1}$$

$$= \{(f_k - 1) - f_{k+1}\} + f_{k+1} \text{ (by induction hypothesis)}$$

$$= (f_k + f_{k+1}) - 1 - f_{k+1}$$

$$= (f_{k+2} - 1) - f_{k+1}$$

This proves the conjecture.

(d)

$$f_0^2 + f_1^2 + \cdots + f_n^2$$

**Answer:**

observe that

$$f_0^2 = 0 = f_0 f_1$$

$$f_0^2 + f_1^2 = 1 = f_1 f_2$$

$$f_0^2 + f_1^2 + f_3^2 = 6 = f_3 f_4$$

So we make the conjecture that

$$f_0^2 + f_1^2 + \cdots + f_n^2 = f_n f_{n+1}.$$

We will prove the conjecture by using induction. We already checked base cases. Assume the conjecture is true for $n = k$. We need to show that the conjecture is true for $n = k + 1$.

$$f_0^2 + f_1^2 + \cdots + f_{k+1}^2$$

$$= \{f_0^2 + f_1^2 + \cdots + f_k^2\} + f_{k+1}^2$$

$$= \{f_k f_{k+1}\} + +f_{k+1}^2 \text{ (by induction hypothesis)}$$

$$= f_{k+1}(f_k + f_{k+1})$$

$$= f_{k+1} f_{k+2}$$

This proves the conjecture.

2. **Brualdi 7.2**

Prove that the $n$th Fibonacci number $f_n$ is the integer which is closest to the number

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

**Answer:**

By Theorem 7.1.1, we have

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Therefore

$$\left| f_n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \right| = \left| \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \right|.$$

Now $2 < \sqrt{5} < 3$, therefore $|1-\sqrt{5}| < 2$, and we get $\frac{|1-\sqrt{5}|}{2} < 1$.

Therefore

$$\left| \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \right|. < \frac{1}{\sqrt{5}} < \frac{1}{2}$$

and the result follows.

3. **Brualdi 7.4**

Let $m$ and $n$ be positive integers. Prove that if $m$ is divisible by $n$, then $f_m$ is divisible by $f_n$.

**Answer:**

First we will prove

$$f_{m+n} = f_{m-1}f_n + f_m f_{n+1}.$$

by induction on $m$. The formula is true when $m = 1$, since

$$f_0 f_n + f_1 f_{n+1} = f_{n+1}$$

Assume the formula is true for $m = k$, then we need to show that the formula is true for $m = k + 1$.

$$f_{k+1-1}f_n + f_{k+1}f_{n+1}$$

$$= f_k f_n + f_{k+1}f_{n+1}$$

$$= f_k f_n + \{f_{k-1} + f_k\}f_{n+1}$$

$$= f_k f_n + f_{k-1}f_{n+1} + f_k f_{n+1}$$

$$= f_k f_n + f_{k-1}\{f_n + f_{n-1}\} + f_k f_{n+1}$$

$$= \{f_k f_n + f_{k-1}f_{n-1}\} + \{f_k f_{n+1} + f_{k-1}f_n\}$$

$$= f_{k+n-1} + f_{k+n} \text{ by induction hypothesis}$$

$$= f_{k+1+n}$$

This proves the formula.

Now we will prove that when $m = nk$ then $f_m$ is divisible by $f_n$ by induction on $k$. When $k = 1$, $f_m = f_n$ and the induction hypothesis is trivially true. Let $f_m$ be divisible by $f_n$ when $m = mk$. We have to show that $f_{n(k+1)}$ is divisible by $f_n$. But

$$f_{n(k+1)} = f_{nk+n}$$

$$= f_{nk-1}f_n + f_{nk}f_{n-1} \text{ (by the formula we proved)}$$

The first term is obviously divisible by $f_n$. The second term is divisible by $f_n$ because $f_{nk}$ is divisible by $f_n$ by induction hypothesis.This implies that $f_{n(k+1)}$ is divisible by $f_n$. This proves the result.

4. **Brualdi 7.23**

Determine the generating function for each of the following sequences.

(a)

$$c^0 = 1, c, c^2, \ldots, c^n \ldots$$

**Answer:**

The generating function is

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots + c^nx^n + \ldots$$

(b)

$$1, -1, 1, -1, \ldots, (-1)^n, \ldots$$

**Answer:**

The generating function is

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^nx^n + \ldots$$

(c)

$$\binom{\alpha}{0}, -\binom{\alpha}{1}, \binom{\alpha}{2}, \ldots, (-1)^n\binom{\alpha}{n}, \ldots$$

**Answer:**

The generating function is

$$(1-x)^\alpha = \binom{\alpha}{0} - \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 - \cdots + (-1)^n\binom{\alpha}{n}x^n + \ldots$$

(d)

$$1, \frac{1}{1!}, \frac{1}{2!}, \ldots \frac{1}{n!}, \ldots$$

**Answer:**

The generating function is

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \ldots$$

(e)

$$1, -\frac{1}{1!}, \frac{1}{2!}, \ldots (-1)^n\frac{1}{n!}, \ldots$$

**Answer:**

The generating function is

$$e^{-x} = 1 - \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + (-1)^n\frac{1}{n!}x^n + \ldots$$

5. **Bruualdi 7.27** Determine the generating function for the sequence of cubes

$$0, 1, 8, \ldots, n^3, \ldots$$

**Answer:**

Let

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \ldots$$

Then

$$f'(x) = \tfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \ldots$$

$$xf'(x) = \tfrac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots + nx^n + \ldots$$

$$(xf'(x))' = \tfrac{2x}{(1-x)^3} + \tfrac{1}{(1-x)^2}$$

$$= 1 + 4x + 9x^2 + 16x^3 + \cdots + n^2x^{n-1} + \ldots$$

$$x(xf'(x))' = \tfrac{2x^2}{(1-x)^3} + \tfrac{x}{(1-x)^2}$$

$$= 1 + 4x^2 + 9x^3 + 16x^4 + \cdots + n^2x^n + \ldots$$

$$(x(xf'(x))')' = \tfrac{6x^2}{(1-x)^4} + \tfrac{6x}{(1-x)^3} + \tfrac{1}{(1-x)^2}$$

$$= 8x + 27x^2 + 64x^3 + \cdots + n^3x^{n-1} + \ldots$$

$$x(x(xf'(x))')' = \tfrac{6x^3}{(1-x)^4} + \tfrac{6x^2}{(1-x)^3} + \tfrac{x}{(1-x)^2}$$

$$= 8x^2 + 27x^3 + 64x^4 + \cdots + n^3x^n + \ldots$$

Therefore the required generating function is

$$\frac{6x^3}{(1-x)^4} + \frac{6x^2}{(1-x)^3} + \frac{x}{(1-x)^2}.$$

6. **Brualdi 7.31**

Determine the generating function for the number $h_n$ of nonnegative integral solutions of

$$2e_1 + 5e_2 + e_3 + 7e_4 = n$$

**Answer:**

The the generating function for the number $h_n$ of nonnegative integral solutions of

$$2e_1 + 5e_2 + e_3 + 7e_4 = n$$

is

$$\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^7}\right)$$

.

Note the powers of $x$ are coefficients of the given equation.

7. **Brualdi 8.1**

   Let $2n$ (equally spaced) points be chosen on a circle. Show that the number of ways to join these points in pairs, so that the resulting $n$ line segments do not intersect, equals the $n$ th Catalan number $C_n$.

   **Answer:**

   Let the number of ways to join $2n$ points on a circle in pairs, so that the resulting $n$ line segments do not intersect be denoted by $a_n$. Choose two points $P$ and $Q$ on the circle. Then we must have even number of points on either side of the line $PQ$ (see Figure 1) for the lines to be non-intersecting.

   Thus for a given side, we can choose 0 points or 2 points or 4 points or $n-2$ points. This leads to a recurrence relation

   $$a_n = a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_{n-1} a_0$$

   where $a_0 = 1$.

   Let $b_n = a_{n-1}, (n \geq 1)$. Then

   $$b_n = b_1 b_{n-1} + b_2 b_{n-2} + \cdots + b_{n-1} b_1, b_1 = 1.$$

   By Theorem 7.6.1, we have

   $$b_n = \frac{1}{n}\binom{2n-1}{n-1}$$

   Thus $a_n = b_{n+1} = \frac{1}{n+1}\binom{2n}{n} = C_n$

8. **Brualdi 8.2** Prove that the number of 2-by-$n$ arrays

   $$\begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ \\ x_{21} & x_{22} & \ldots & x_{2n} \end{bmatrix}$$

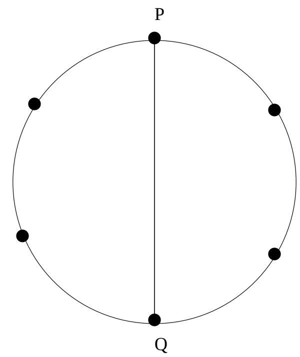


Figure 1: Problem 8.1

that can be made from the numbers $1, 2, \ldots, 2n$ so that

$$x_{11} < x_{12} < \cdots < x_{1n},$$

$$x_{21} < x_{22} < \cdots < x_{2n},$$

and

$$x_{11} < x_{21}, x_{12} < x_{22} \ldots x_{1n} < x_{2n}$$

equals the Catalan number, $C_n$.

**Answer:** Consider a 2-by-$n$ array and let consider the corresponding sequence $a_1, a_2, \ldots, a_{2n}$ consisting of $\pm 1$, obtained by taking $a_j = 1$, if $j$ is in the first row and $a_j = -1$, if $j$ is in the second row. Thus, an array satisfying the given conditions corresponds to sequences of $a_i = \pm 1$ with

$$\sum_{i=0}^{k} a_i \geq 0, (k = 0, 1, 2, \ldots, 2n)$$

The number of such sequences, by Theorem 8.1.1 equals the Catalan number and the result follows.

9. **Brualdi 8.3**

Write out all the multiplication schemes for four numbers and the triangularization of a convex polygonal region of five sides corresponding to them.

**Answer:**

Please refer to page 258 for the list of multiplication schemes for $n = 3$. Every scheme extends into a multiplication of four numbers in 10 ways. For example the multiplication scheme

$$(a_1 \times (a_2 \times a_3))$$

gives the following 10 schemes:

$$(a_4 \times (a_1 \times (a_2 \times a_3)))$$

$$((a_1 \times (a_2 \times a_3)) \times a_4)$$

$$((a_1 \times a_4) \times (a_2 \times a_3))$$

$$((a_4 \times a_1) \times (a_2 \times a_3))$$

$$(a_1 \times (a_4 \times (a_2 \times a_3)))$$

$$((a_4 \times (a_1 \times (a_2 \times a_3)))$$

$$(a_1 \times ((a_2 \times a_3) \times a_4))$$

$$(a_1 \times (a_4 \times (a_2 \times a_3)))$$

There are a total of 120 multiplication schemes for $n = 4$ since there are 12 multiplication schemes for $n = 3$. (Have fun listing them in your hws.) Every multiplication scheme of 4 numbers correspond to a triangulation of a polygon with 5 vertices. A few examples are shown in Figure 2.

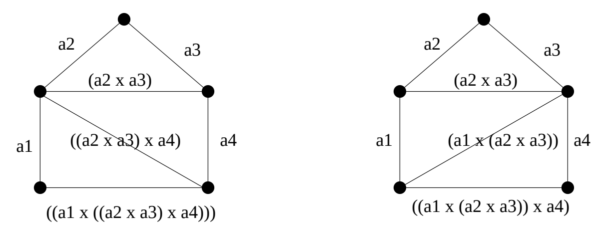


Figure 2: Problem 8.3: Triangulations corresponding to multiplication schemes

10. **Bruualdi 8.4**

Determine the triangularization of a convex polygonal region corresponding to the following multiplication schemes:

(a)

$$(a_1 \times (((a_2 \times a_3) \times (a_4 \times a_5)) \times a_6))$$

**Answer:**

(b)

$$(((a_1 \times a_2) \times (a_3 \times (a_4 \times a_5)) \times ((a_6 \times a_7) \times a_8)))$$

**Answer:**

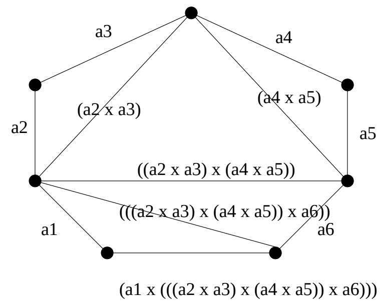


Figure 3: Problem 8.4 (a)

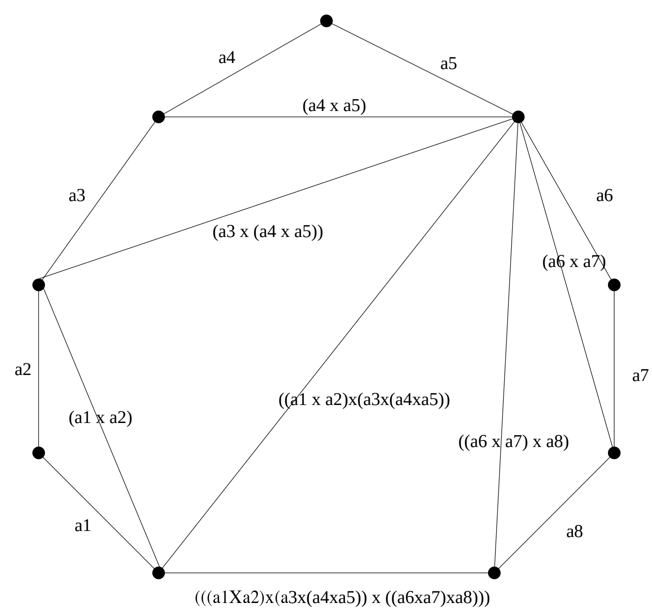


Figure 4: Problem 8.4 (b)