MAT 145: Homework Solutions #8

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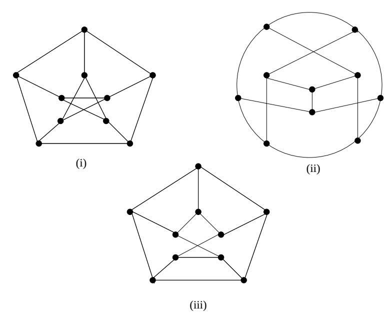
1. **Brualdi 11.12**

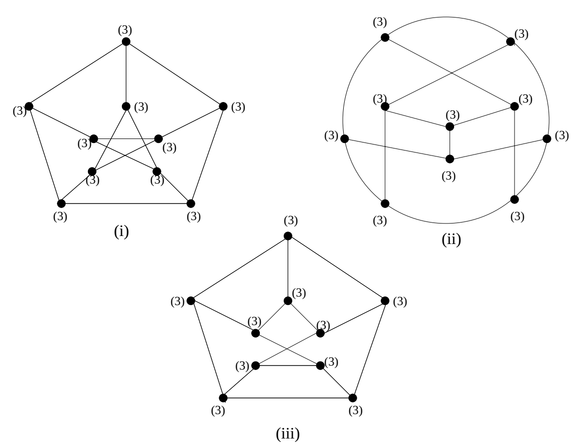
   Determine which pairs of the multi-graphs in Figure 11.40 are isomorphic and if isomorphic, find an isomorphism.

Figure 1: Problem 11.12 Figure 11.40



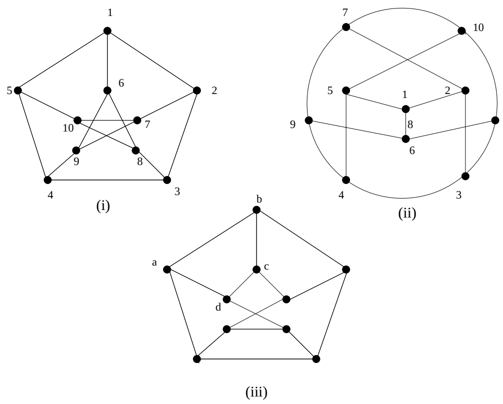
Answer:

Figure 2: Problem 11.12 Check for degree differences. Note that the degree sequences of vertices in all the graphs are same.



Graph (i) and (ii) are isomorphic. A relabelling of vertices that shows the isomorphism is shown in the figure below. The graph (iii) is not isomorphic since it contains a 4-cycle a-b-c-d, and the graphs (i) and (ii) have no 4-cycles.

Figure 3: Problem 11.12 Isomorphism of graphs



2. **Brualdi 11.20**

Prove that a graph of order n with at least

$$\frac{(n-1)(n-2)}{2} + 1$$

edges must be connected. Give an example of a disconnected graph of order n with one fewer edge.

Answer: [By M. Mulase]

We note that the complete graph has the largest number of edges among the graphs with the same order. A graph of order n (i.e., a graph that has n vertices) is *disconnected* if it has two subgraphs of orders k and $n-k$ that are not connected, here k is some integer in the range $1 \leq k \leq n-1$. Among all such graphs, the largest number of edges is achieved if we have the complete graph of order k and the complete graph of order $n-k$. If you add just one more edge to this situation, then the added edge should connect the two disjoint components to make a connected graph of order n. Thus we have to establish an inequality that

$$\binom{k}{2} + \binom{n-k}{2} < \frac{(n-1)(n-2)}{2} + 1$$

for every $1 \leq k \leq n-1$. [**Note:** It is a very common mistake that you prove this inequality only for $k = 1$ and think you are done. Of course that is not the right answer!]

The proof goes as follows. First we notice that the inequality we want to establish is equivalent to

$$\binom{k}{2} + \binom{n-k}{2} \leq \frac{(n-1)(n-2)}{2},$$

or

$$k(k-1) + (n-k)(n-k-1) \le (n-1)(n-2) \ .$$

To prove this, we consider the function

$$f(k) = (n-1)(n-2) - k(k-1) - (n-k)(n-k-1) \ ,$$

where k is a variable and n is a fixed constant. By an algebraic computation, we obtain

$$f(k) = 2(-k^2 + nk - n + 1) = 2\big(k(n-k) - (n-1)\big) \ .$$

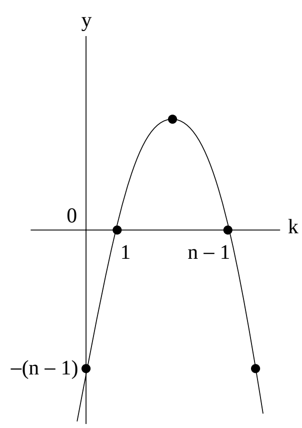


Figure 4: $y = k(n-k) - (n-1)$.

From Figure 4, we see that if $1 \le k \le n-1$, then $f(k) \ge 0$. This implies that

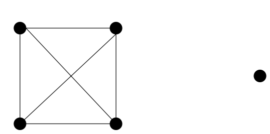
$$\binom{k}{2} + \binom{n-k}{2} < \frac{(n-1)(n-2)}{2} + 1$$

for the same range of k, which establishes the claim.

In class I gave a slightly different explanation, by *removing* edges from a complete graph. That method leads to exactly the same quadratic inequality a lot faster. The proof I give here seems to be a little easier to understand, though.

An example of a disconnected graph of order 5 with only 6 edges is shown in the figure below.

Figure 5: Problem 11.20 Example of disconnected graph



3. **Brualdi 11.21**

   Let G be a general graph with exactly two vertices x and y of odd degree. Let G^* be the general graph obtained by putting a new edge $\{x, y\}$ joining x and y. Prove that G is connected if and only if G^* is connected.

   Answer: [by M. Mulase]

   Clearly if G is connected, then G^* is connected.

   Conversely, assume that G^* is connected. We note that every vertex of G^* has an even degree. Notice that G is obtained by *removing* an edge connecting x and y. Now suppose G is disconnected. Then the vertices x and y should belong to different connected components of G. Since all other vertices have even degrees, the total degree of each connected component becomes odd. But the total degree (i.e., the sum of the degrees of all vertices) is *always* even for any general graph. Thus it is impossible for x and y to belong to different connected components. Which means removing an edge from a connected general graph with even degree vertices everywhere does not affect the connectivity. Therefore, G is connected. $\square$

   Remark: The above argument is used when you prove the criterion for the existence of an Eulerian trail by induction on the number of edges, instead of the way the book does. I strongly recommend you try the proof!!!

4. **Brualdi 11.27**

   Determine the adjacency matrices of the first and second multi-graphs in Figure 11.40.

   Answer:

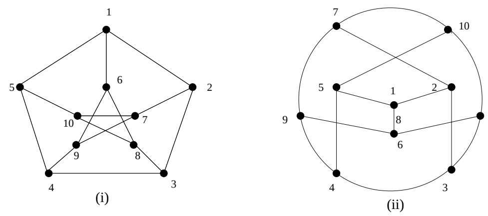
   Label the graphs as shown in Figure 6. Since the two graphs are isomorphic, they have the same adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

5. **Brualdi 11.28**

   Let A and B be two $n \times n$ matrices of numbers whose entries are denoted by a_{ij} and $b_{ij}, (1 \leq i, j \leq n)$, respectively. Define the product $A \times B$ to be the $n \times n$ matrix C

Figure 6: Problem 11.27



whose entry c_{ij} in row i and column j is given by

$$c_{ij} = \sum_{p=1}^{n} a_{ip} b_{pj} (1 \leq i, j \leq n).$$

If k is a positive integer define

$$A^k = A \times A \times A \times \cdots \times A(kA's).$$

Now let A denote the adjacency matrix of a general graph of order n with vertices $a_1, a_2, \ldots, a_n$. Prove that the entry in row i, column j of A^k equals the number of walks of length k in G joining vertices a_i and a_j.

Answer: Proof by induction:

Since A is an adjacency graph, A_{ij} denotes the number of edges joining vertices a_i and a_j. Thus A_{ij} is the number of walks of length 1 joining vertices a_i and a_j.

Assume that the entry in row i, column j of A^k equals the number of walks of length k in G joining vertices a_i and a_j. We need to show that the result holds for $n = k + 1$. Since $A^{k+1} = A^k A$, we have

$$A_{ij}^{k+1} = \sum_{p=1}^{n} A_{ip}^k A_{pj} (1 \leq i, j \leq n).$$

Consider a term in the summand, $A_{ip}^k A_{pj}$. By the induction hypothesis, A_{ip}^k denotes the number of walks of length k between the vertices a_i and a_p and A_{pj} denotes the number of edges between vertices a_p and a_j. Thus we get, $A_{ip}^k A_{pj}$ walks of length $k+1$ between the vertices a_i and a_j by extending the walks of length k between vertices a_i and a_p by the edges between vertices a_p and a_j. Since every walk of length $k+1$ from a_i to a_j goes through a vertex, say a_p, after k steps, the summation above counts all distinct walks of length $k+1$.

The result follows by induction.

6. **Brualdi 11.36**

Let G be a connected graph. Let γ be a closed walk which contains each edge of G at least once. Let G^* be the multi-graph obtained from G by increasing the multiplicity of each edge from 1 to the number of times it occurs in γ. Prove that γ is a closed Eulerian trail in G^*. Conversely suppose we increase the multiplicity of some of the edges of G and obtain a multi-graph with m edges each of whose vertices has even degree. Prove that there is a closed walk in G of length m which contains each edge of G at least once. (This exercise shows that the Chinese postman problem for G is equivalent to determining the smallest number of copies of the edges of G to be inserted so as to obtain a multi-graph all of whose vertices have even degree.

Answer:

($\Rightarrow$)

Let γ be a closed walk which contains each edge of G at least once. Let G^* be the multi-graph obtained from G by increasing the multiplicity of each edge from 1 to the number of times it occurs in γ. Then γ is a closed Eulerian trail in G^*, because the repeated edges in the walk have become edges in the new graph and γ includes all the edges of the graph G^* exactly once. Of course γ is a closed trail since it starts and ends at the same point.

($\Leftarrow$)

Increase the multiplicity of some of the edges of G and obtain a multi-graph with m edges each of whose vertices has even degree. By Theorem 11.2.2, this multi-graph has an Eulerian circuit β. β is a closed walk in G of length m which contains each edge of G at least once.

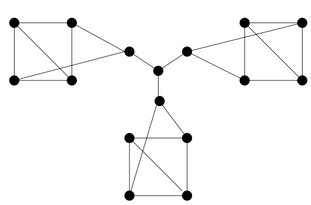
□

7. **Brualdi 11.39**

Call a graph *cubic* if each vertex has degree equal to 3. The complete graph K_4 is the smallest example of a cubic graph. Find an example of a connected, cubic graph that does not have a Hamilton cycle.

Answer: See Figure 7.

Figure 7: Problem 11.39: A cubic graph that does not have a Hamilton cycle



8. **Brualdi 11.47**

   Prove that a bipartite multi-graph with an odd number of vertices does not have a Hamilton cycle.

   Answer: [by M. Mulase]

   Suppose a bipartite multigraph B has a Hamiltonian cycle h. We align all vertices in two columns such that every edge connects a vertex in the left column with a vertex in the right column. Label all vertices of the bipartite graph B following the Hamiltonian cycle h: namely, call the first vertex 1, then follow the cycle h to label the second vertex 2, etc, up to n, and finally comes back to 1. Let us choose the first vertex on the left column. Then every vertix on the left column is labeled with an odd number and every vertix on the right column is labeled with an even number (see Figure 8). Since n is on the right column because it is the last vertex before coming back to 1, it must be an even number. Therefore, the Hamiltonian cycle we are dealing with must have even number of vertices.

   This proves that a bipartite multigraph with an odd number of vertices cannot have a Hamilton cycle.

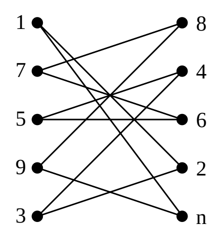


Figure 8: A Hamiltonian cycle on a bipartite graph.

9. **Brualdi 11.56** Grow all the non-isomorphic trees of order 7.

   Answer: Growing trees is indeed a difficult task! Here is the correct list. The previous one had an error, pointed out by our Reader.

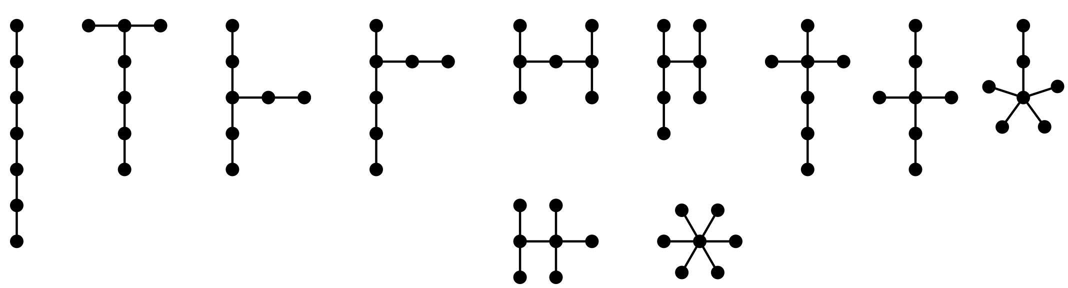


Figure 9: Trees of order 7.

10. **Brualdi 11.57**

Let $(d_1, d_2, \ldots, d_n)$ be a sequence of integers. (a) Prove that there is a tree of order n with this degree sequence iff $d_1, d_2, \ldots, d_n$ are positive integers with sum $d_1 + d_2 + \cdots + d_n = 2(n-1)$.

Answer: [by M. Mulase]

First we have to understand the statement of the claim. Let e be the number of edges of any general graph of order n (order = number of vertices). Then we have

$$d_1 + d_2 + \cdots + d_n = 2e\ . \tag{1}$$

Therefore, the formula we are given is simply saying that $e = n - 1$. Thus the claim we have to prove is that a connected general graph of order n is a tree if and only if its number of edges is equal to $n - 1$. We refer the proof to Chapter 11.5 of the book.

(b) Write an algorithm, which starting with a sequence $(d_1, d_2, \ldots, d_n)$ of positive integers, either constructs a tree with this degree sequence or concludes that none is possible.

Answer: [by M. Mulase]

First we check if $d_1 + d_2 + \cdots + d_n = 2(n-1)$. If yes, proceed. If not, then there is no tree of the degree sequence $(d_1, d_2, \ldots, d_n)$.

Now we have a sequence satisfying the condition

$$d_1 + d_2 + \cdots + d_n = 2(n-1), \qquad d_1 \le d_2 \le \cdots \le d_n\ .$$

If $n = 1$, then the tree consists of a single vertex with no edge. So we assume $n \ge 2$. We start with the n-th vertex, the vertex with the largest degree. For simplicity of presentation, we assume that exactly first i terms of the degree sequence is 1:

$$1 = d_1 = d_2 = \cdots = d_i < d_{i+1} \le d_{i+2} \le \cdots d_n\ .$$

Step 1. Attach $d_n - 1$ edges to the vertex n and also $d_n - 1$ vertices to the end of these attached edges.

Step 2. Attach $d_{n-1} - 2$ edges to the vertex $n - 1$ with degree 1 vertex at the end. Then connect the vertex $n - 1$ to the vertex n with an edge. (See Figure 10.)

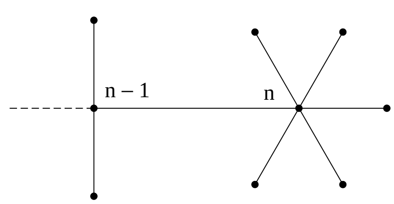


Figure 10: Constructing a tree of a prescribed degree sequence.

Step 3. Repeat Step two with the vertex $n-2$, $n-3$, etc.

Final Step. When all vertices of degree 2 or more are exhausted at vertex $i+1$, attach a single edge with a degree one vertex at the end to the vertex $i+1$.

Why does this procedure work? Let's calculate the number of edges. The degree condition

$$d_1 + \cdots + d_n = 2(n-1)$$

implies

$$d_{i+1} + d_{i+1} + \cdots + d_n = 2(n-1) - i.$$

The procedure shows that the number of vertices of degree greater than 1 is $n-i$, and the number of edges we created is

$$\begin{aligned} d_n + (d_{n-1} - 1) + (d_{n-2} - 1) + \cdots + (d_{i+1} - 1) &= d_{i+1} + d_{i+2} + \cdots d_n - (n-i-1) \\ &= 2(n-1) - i - (n-i-1) \\ &= n-1 \ . \end{aligned}$$

Therefore, the "algorithm" created a connected graph of order n with $n-1$ edges, hence it is a tree!