MAT 16C: SHORT CALCULUS
FIRST MIDTERM EXAMINATION

DATE AND TIME: FRIDAY, APRIL 23, 2004. 11:00–11:50.
ROOM: 202 WELLMAN
INSTRUCTOR: M. MULASE

Name: (Last) (First)

Student ID Number: – –

Remark.

(1) The exam set consists of 5 pages, including the cover sheet.
(2) **Do not de-staple the set.**
(3) **No calculators are allowed in the exam.**
(4) This is a closed-book exam. Do not open the textbook or lecture notes during the exam.
(5) Use the back side of the page if you need scratch paper.

Scores:

Page 2: ____________/8

Page 3: ____________/8

Page 4: ____________/7

Page 5: ____________/2

Total: ______________/25

Problem 1 (8 points). *Let us consider the function*

$$w(x,y,z) = \ln\sqrt{x^2+y^2+z^2}\,.$$

(1) Find the first partial derivative $w_z = \frac{\partial w}{\partial z}$ and give your answer in the simplest form.

Work (1 point): Since $w(x,y,z) = \frac{1}{2}\ln(x^2+y^2+z^2)$, we compute

$$w_z = \frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2z = \frac{z}{x^2+y^2+z^2}.$$

Answer (1 point):

$$w_z = \frac{z}{x^2+y^2+z^2}.$$

(2) Find the value of the second partial derivative $w_{xy} = \frac{\partial^2 w}{\partial x \partial y}$ at the point $(0,0,1)$.

Work (2 points):

$$w_{xy} = \frac{\partial w_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2+z^2}\right) = -\frac{2xy}{(x^2+y^2+z^2)^2}.$$

Therefore, $w_{xy}(0,0,1) = -\frac{0}{1} = 0$.

Answer (1 point):

$$w_{xy}(0,0,1) = \underline{\qquad\qquad 0 \qquad\qquad}.$$

(3) The collection of all points (x,y,z) that satisfy the equation

$$w(x,y,z) = 0$$

forms a sphere in the space. Find its center and the radius.

Work (1 point): We note that

$$w(x,y,z) = \ln\sqrt{x^2+y^2+z^2} = 0 \Longleftrightarrow \sqrt{x^2+y^2+z^2} = 1.$$

Therefore, the equation of the sphere is

$$x^2+y^2+z^2 = 1.$$

Answer (2 points):

Center = $(0,0,0)$ Radius = $\underline{\quad 1 \quad}$.

Problem 2 (8 points. In this problem the answer you obtain in each step will affect the next step. Please proceed with great care). *Let us consider the differential equation*

$$y' + xy = 0$$

for an unknown function $y = y(x)$.

(1) Find the general solution of the above differential equation.

Work (2 points): This is an equation solved by separation of variables.

$$y' = -xy \Longleftrightarrow \frac{dy}{y} = -xdx \Longleftrightarrow \int \frac{dy}{y} = \int -xdx \Longleftrightarrow \ln y = -\frac{x^2}{2} + c.$$

Therefore,

$$y = e^{-\frac{x^2}{2}+c} = Ce^{-\frac{x^2}{2}}.$$

Answer (1 point):

$$y = Ce^{-\frac{x^2}{2}}.$$

(2) Find the solution to the initial condition $y = 1$ when $x = 0$.

Work (2 points):

$$1 = Ce^{-\frac{0}{2}} = C.$$

Answer (1 point): $y = e^{-\frac{x^2}{2}}$.

(3) Find the values of x at which the particular solution you found in the problem above attains the maximum value.

Work (1 point): Since $x^2 \geq 0$, the range of $y = e^{-\frac{x^2}{2}}$ is $0 < y \leq 1$. From the differential equation we know $y' = -xy$ and $y' = 0$ only when $x = 0$. Therefore, y is maximum at $x = 0$, with the maximum value 1.

Answer (1 point): $x = \underline{\quad 0 \quad}$.

The guidelines for solving a linear differential equation are the following:

(1) Write the equation in standard form

$$y' + P(x)y = Q(x).$$

(2) Find the integrating factor

$$u(x) = e^{\int P(x)dx}.$$

(3) Evaluate the following integral to find the general solution

$$y = \frac{1}{u(x)} \int Q(x)u(x)dx \ .$$

Problem 3 (7 points)**.** *Follow the above guidelines to find the general solution of the linear differential equation*

$$y' = x - y \ .$$

(1) Find the integrating factor.

Work (1 point): The standard form of the equation is $y' + y = x$. Hence $P(x) = 1$, and we have

$$u(x) = e^{\int 1dx} = e^x.$$

Answer (1 point): $u(x) = e^x$.

(2) Find the general solution:

Work (3 point): Since $Q(x) = x$, we compute

$$y = e^{-x} \int xe^x dx = e^{-x} \left(xe^x - e^x + C\right) = x - 1 + Ce^{-x}.$$

Answer (2 point): $y = x - 1 + Ce^{-x}$.

Problem 4 (2 points)**.** *Consider the surface defined by the equation*

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1 \ .$$

(1) What is the name of this surface?

Answer (1 point): *Ellipsoid*! .

(2) Find the equation for the xz-trace of the surface.

Answer (1 point): You take $y = 0$ to find the xz-trace. Therefore,

$$\frac{x^2}{4} + z^2 = 1 \ .$$