MAT 16C: SHORT CALCULUS
SECOND MIDTERM EXAMINATION WITH SOLUTIONS

DATE AND TIME: FRIDAY, MAY 14, 2004. 11:00–11:50.
ROOM: 202 WELLMAN
INSTRUCTOR: M. MULASE

Name: (Last) (First)

Student ID Number: – –

Remark.

1. The exam set consists of 5 pages, including the cover sheet.
2. **Do not de-staple the set.**
3. **No calculators are allowed in the exam.**
4. This is a closed-book exam. Do not open the textbook or lecture notes during the exam.
5. Use the back side of the page if you need scratch paper.

Scores:

Page 2: ____________/6

Page 3: ____________/6

Page 4: ____________/7

Page 5: ____________/6

Total: ______________/25

Problem 1 (6 points)**.** *Solve each of the following problems using double integrals.*

(1) Find the average of the function $f(x, y) = \frac{2y}{x^2+y^2}$ over the plane region R that is bounded by $y = x, y = 2x$, and $x = 2$.

Work (2 points): The region R has a triangular shape connecting $(0, 0)$, $(2, 2)$ and $(2, 4)$. Since its base length is 2 and height is 2, the area of R is $1/2 \times 2 \times 2 = 2$. You have to choose the right order of the double integral to compute the average. We integrate by y first, which is bound by $x \leq y \leq 2x$, and then integrate by x with bound $0 \leq x \leq 2$. Let $u = x^2 + y^2$. Holding x as constant, we have $du = 2ydy$. Hence

$$\frac{1}{2} \iint_R \frac{2y}{x^2+y^2}\, dA = \frac{1}{2} \int_0^2 \left(\int_x^{2x} \frac{2y}{x^2+y^2}\, dy \right) dx = \frac{1}{2} \int_0^2 \left(\int_{2x^2}^{5x^2} \frac{du}{u} \right) dx$$

$$= \frac{1}{2} \int_0^2 \Big[\ln u \Big]_{2x^2}^{5x^2} dx = \frac{1}{2} \int_0^2 \left(\ln(2x^2) - \ln(5x^2) \right) dx$$

$$= \frac{1}{2} \int_0^2 \ln\left(\frac{5}{2}\right) dx = \frac{1}{2} \ln\left(\frac{5}{2}\right) [x]_0^2 = \ln\left(\frac{5}{2}\right).$$

Answer (1 point): Average $= \ln\left(\frac{5}{2}\right)$.

(2) Find the volume of the solid that is bounded by the graphs of the equations $x = 0, y = 0, z = 0$, and $x + y + z = 1$.

Work (2 points): The shape of the solid is a triangular pyramid placed on its side. We integrate $z = 1 - x - y$ on the triangular region on the xy-plane bound by $x = 0$, $y = 0$, and $x + y = 1$. The order of integration does not matter. For a choice of y in $0 \leq y \leq 1$, $x = 1 - y$.

$$\int_0^1 \int_0^{1-y} (1 - x - y)\, dxdy = \int_0^1 \left[(1-y)x - \frac{x^2}{2} \right]_0^{1-y} dy$$

$$= \int_0^1 \left((1-y)^2 - \frac{(1-y)^2}{2} \right) dy = \int_0^1 \left(\frac{(1-y)^2}{2} \right) dy = -\int_1^0 u^2/2du = \frac{1}{6}.$$

Answer (1 point): Volume $= \frac{1}{6}$.

Problem 2 (6 points). *Evaluate each of the following double integrals.*

(1)
$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x+y)\ dy\ dx.$$

Work (2 points): Let $u = 1 - x^2$. Then $du = -2xdx$.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x+y)\ dy\ dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left(x\sqrt{1-x^2} + \frac{1}{2}(1-x^2)\right) dx = -\frac{1}{2}\int_1^0 \sqrt{u} du + \int_0^1 \frac{1}{2}(1-x^2) dx$$

$$= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} = \frac{2}{3}.$$

Answer (1 points)

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x+y)\ dy\ dx = \frac{2}{3}.$$

(2)
$$\int_{-1}^1 \int_{-1}^1 (x^2+y^2)\ dx\ dy.$$

Work (2 points):

$$\int_{-1}^1 \int_{-1}^1 (x^2+y^2)\ dx\ dy = \int_{-1}^1 \left[\frac{x^3}{3} + xy^2\right]_{-1}^1 dy = \int_{-1}^1 \left(\frac{2}{3} + 2y^2\right) dy$$

$$= \left[\frac{2}{3}\ y + \frac{2}{3}\ y^3\right]_{-1}^1 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}.$$

Answer (1 points)

$$\int_{-1}^1 \int_{-1}^1 (x^2+y^2)\ dx\ dyx = \frac{8}{3}.$$

Problem 3 (4 points)**.** *Let x, y, z be positive numbers. Find extremum of $f(x,y,z) = x^2 + y^2 + z^2$ under the constraint $x + 2y + 3z = 1$.*

Work (3 points): The constraint function is $g(x,y,z) = x+2y+3z-1$. Let

$$F(x,y,z,\lambda) = x^2 + y^2 + z^2 - \lambda(x + 2y + 3z - 1).$$

We find $F_x = 2x - \lambda$, $F_y = 2y - 2\lambda$, $F_z = 2z - 3\lambda$, and $F_\lambda = -(x+2y+3z-1)$. Equating all partials equal to 0, we obtain $x = 1/2\lambda$, $y = \lambda$, $z = 3/2\lambda$, and the final equation becomes

$$\frac{1}{2}\lambda + 2\lambda + 3 \times \frac{3}{2}\lambda = 7\lambda = 1.$$

Thus we find $\lambda = \frac{1}{7}$. Putting it back to the equations, we find

$$x = \frac{1}{14}, y = \frac{1}{7}, z = \frac{3}{14}.$$

Answer (1 point): The extremum is attained at $\left(\frac{1}{14}, \frac{1}{7}, \frac{3}{14}\right)$, and the extremal value is $\frac{1}{14}$.

Note: The value thus established in only an extremal value.

Problem 4 (3 points)**.** *Find the critical point of the function*

$$f(x,y) = x^2 - 2y^2.$$

Examine if it is a relative minimum, a relative maximum, or a saddle point.

Work (1 point): We compute $f_x = 2x$, $f_y = -4y$, $f_{xx} = 2$, $f_{yy} = -4$, and $f_{xy} = 0$. Therefore, the only critical point is at $x = 0, y = 0$, or at the origin $(0,0)$, by the first partials test. The second partials test shows that since

$$d = f_{xx} \cdot f_{yy} - \left(f_{xy}\right)^2 = -8 < 0,$$

the critical point is a saddle point.

Answer (2 point): Critical Point = $(0,0)$. It is a saddle point.

Problem 5 (6 points)**.** *For each of the following sequences, determine if it is convergent or divergent, and if convergent, give the limit value as n goes to infinity.*

Example: $a_n = n$. <u>Divergent.</u> $\lim_{n\to\infty} a_n =$ <u>N/A</u>.

(3) $a_n = \dfrac{(n-1)(n+1)}{n^2+1}$. <u>Convergent.</u> $\lim\limits_{n\to\infty} a_n =$ <u>1</u>.

(4) $a_n = \dfrac{n!}{(n-1)!}$. <u>Divergent.</u> $\lim\limits_{n\to\infty} a_n =$ <u>N/A</u>.

(5) $a_n = \dfrac{\sqrt{n}+\sqrt{n-1}}{\sqrt{n+1}}$. <u>Convergent.</u> $\lim\limits_{n\to\infty} a_n =$ <u>2</u>.

(6) $a_n = (-1)^n \dfrac{n^2-1}{n^2+2n+1}$. <u>Divergent.</u> $\lim\limits_{n\to\infty} a_n =$ <u>N/A</u>.

(7) $a_n = (-1)^{n-1}\dfrac{1}{2^n}$. <u>Convergent.</u> $\lim\limits_{n\to\infty} a_n =$ <u>0</u>.

(8) $a_n = \left(1+\dfrac{1}{n}\right)^{2n}$. <u>Convergent.</u> $\lim\limits_{n\to\infty} a_n =$ <u>e^2</u>.