MAT 21B: CALCULUS
FIRST MIDTERM EXAMINATION

DATE AND TIME: FRIDAY, JANUARY 26, 2001. 9:00A.M.-9:50A.M.
ROOM: 2205 HARING
INSTRUCTOR: M. MULASE

Name: (Last) ______________________ (First) ______________

Student ID Number: ______ – ______ – ______

CRN Number: ______________________

Remark.

1. The exam consists of 5 pages, including the cover sheet.
2. **Do not de-staple the exam.**
3. It is an open-book exam. Thus you can use anything written in the textbook and your lecture notes.

Scores:

Page 2: ____________/4

Page 3: ____________/4

Page 4: ____________/3

Page 5: ____________/5

Total: ____________/16

Problem 1 (4 points)**.**

1. Evaluate

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2.$$

   (a) Work (You can use whatever the formula you know):

   Solution. We use the formula $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$. Thus

$$(1/n^3) \sum_{i=1}^{n} i^2 = \frac{1}{6} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n}.$$

   As $n \to \infty$, we have the limit $2/6 = 1/3$.

   (b) Answer:

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{3}.$$

2. Evaluate

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}$$

   using the telescopic method.

   Solution.

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$
$$= \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}.$$

3. Find

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i(i+1)}.$$

   Solution.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i(i+1)} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Problem 2 (4 points)**.**

1. Find the antiderivative
$$\int \sin^3(x)\cos(x)dx.$$
   (a) Work:

   Solution. Let $u = \sin(x)$. Then $du = \cos(x)dx$. Therefore,
$$\int \sin^3(x)\cos(x)dx = \int u^3 du$$
$$= \frac{1}{4}u^4 + C = \frac{1}{4}\big(\sin(x)\big)^4 + C.$$
   (b) Answer:
$$\frac{1}{4}\big(\sin(x)\big)^4 + C.$$
   If you forgot to put C, that's all right. But $1/4 \cdot u^4 + C$ is **not** an acceptable answer.

2. Evaluate
$$\int_{-\pi}^{\pi}\big(x + x^3 + x^5 + x^7 + x^9 + x^{11} + \sin(x)\big)dx.$$
   (a) Work, or if you don't want to work, then provide a reasoning of your answer:

   Solution. The integrand is an odd function. Let
$$f(x) = x + x^3 + x^5 + x^7 + x^9 + x^{11} + \sin(x).$$
   Then $f(-x) = -f(x)$. Therefore,
$$\int_{-\pi}^{\pi} f(x)dx = 0.$$
   (b) Answer:
$$\int_{-\pi}^{\pi}\big(x + x^3 + x^5 + x^7 + x^9 + x^{11} + \sin(x)\big)dx = 0.$$

Problem 3 (3 points)**.** Consider the planar region enclosed by the graph of $y = \sqrt{\sin(x)}$ on the interval $[0, \pi]$ and the x axis. If we rotate this planar region about the x-axis, then we obtain a solid that looks like a football.

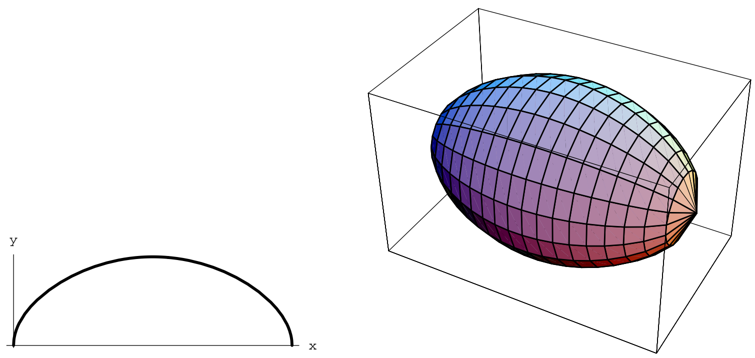

FIGURE 1. The solid enclosed in a surface of revolution that is obtained by rotating the graph of $y = \sqrt{\sin(x)}$ about the interval $[0, \pi]$ of the x-axis.

1. Express the volume of this solid as a definite integral.

   Solution. We use the potato chip method.

$$\text{Volume} = \int_0^\pi \pi\left(\sqrt{\sin(x)}\right)^2 dx.$$

2. Compute the volume of this solid.
   (a) Work:

   Solution.

$$\begin{aligned}\text{Volume} &= \int_0^\pi \pi\left(\sqrt{\sin(x)}\right)^2 dx \\ &= \int_0^\pi \pi \sin(x) dx \\ &= \pi\left(-\cos(x)\right)\Big|_0^\pi \\ &= \pi(1+1) = 2\pi.\end{aligned}$$

   (b) Answer:
   The volume of the solid is 2π.

Problem 4 (5 points)**.** Consider a function $y = f(x)$ defined for $x \geq 0$ that satisfies $f(x) > 0$ and an equation

$$\int_0^x f(t)dt = 1 - \sqrt{f(x)}.$$

1. Find $f(0)$.

   Solution. Substituting $x = 0$ in the above equation, we obtain $0 = \int_0^0 f(t)dt = 1 - \sqrt{f(0)}$. Therefore, $f(0) = 1$.

2. Derive a differential equation that $f(x)$ satisfies.

   Solution. Differentiating the original equation, we obtain

   $$f(x) = -\frac{1}{2}\big(f(x)\big)^{-\frac{1}{2}} f'(x).$$

3. Find $f'(0)$.

   Solution. Since $f(0) = 1$, we have

   $$1 = f(0) = -\frac{1}{2}\big(f(0)\big)^{-\frac{1}{2}} f'(0) = -\frac{1}{2} f'(0).$$

   Hence $f'(0) = -2$.

4. Find $f(x)$ as a function in x.
   (a) Work:

   Solution. From 2 above, we have

   $$1 = -\frac{1}{2}\big(f(x)\big)^{-\frac{3}{2}} f'(x).$$

   Integrating the above equation, we have

   $$x + C = \int -\frac{1}{2}\big(f(x)\big)^{-\frac{3}{2}} f'(x)dx = \big(f(x)\big)^{-\frac{1}{2}}.$$

   Since $f(0) = 1$, we know $C = 1$.

   (b) Answer:

   $$f(x) = \frac{1}{(x+1)^2}.$$