MAT 22A: LINEAR ALGEBRA
FINAL EXAMINATION

DATE AND TIME: TUESDAY, MARCH 23, 2004. 8:00–10:00
ROOM: 6 WELLMAN AND 202 WELLMAN
INSTRUCTOR: M. MULASE

Name: (Last) ____________ (First) ____________

Student ID Number: ____ – ____ – ____

Signature: ____________

Remark.

(1) You are required to abide by the *UCD Code of Academic Conduct.*
(2) The exam set consists of 6 pages, including the cover sheet.
(3) **Do not de-staple the set.**
(4) This is a closed-book exam. Do not open the textbook or lecture notes during the exam.
(5) You have full two hours.
(6) Use the back side of the sheet if you need scratch paper.

IMPORTANT NOTICE:

THE FINAL EXAMINATION ROOM HAS BEEN CHANCED.

(1) **IF THE INITIAL OF YOUR LAST NAME IS A - L, PLEASE GO TO ROOM 006 WELLMAN.**
(2) **IF THE INITIAL OF YOUR LAST NAME IS M - Z, PLEASE GO TO ROOM 202 WELLMAN.**

Problem 1 (3 points)**.** *Find the determinant of* $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

Work (2 points):

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & -1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{vmatrix} = 2 \times (-2) \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Answer (1 point): $\det A =$ ___-4___.

Problem 2 (2 points)**.** *For each of the following statements, determine if it is true or false, and circle the correct answer.*

(1) Let A be an $n \times n$ matrix. If A is a symmetric matrix, then it always has n distinct real eigenvalues. **True False**.

The statement is false because the identity matrix I_n is symmetric but its only eigenvalue is 1.

(2) Let $L : V \longrightarrow W$ be a linear transformation. If L is one-to-one, then the kernel of L consists only of the zero vector $\mathbf{0}$. **True False**.

The statement is true. We first note that $L(\mathbf{0}) = \mathbf{0}$. The transformation L being one-to-one means there is only one vector that is sent to $\mathbf{0} \in W$. Therefore, $\mathrm{Ker}(L) = \{\mathbf{0}\}$, as stated.

In the above problem you are not required to give any proof or reason.

Problem 3 (5 points)**.** *Consider the* 3×3 *matrix* $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$, *and answer the following questions.*

(1) Find the characteristic polynomial of A, and express it as a polynomial in λ in the fully expanded form.
Work (1 point):

$$\begin{vmatrix} \lambda - 2 & -2 & -2 \\ -2 & \lambda - 2 & -2 \\ -2 & -2 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 - 8 - 8 - 4(\lambda - 2) - 4(\lambda - 2) - 4(\lambda - 2)$$

$$= \lambda^3 - 6\lambda^2 + 12\lambda - 8 - 16 - 12\lambda + 24 = \lambda^3 - 6\lambda^2 = \lambda^2(\lambda - 6).$$

Answer (1 point): $\underline{\lambda^3 - 6\lambda^2}$.

(2) List all eigenvalues of A (1 point): $\lambda = \underline{\quad 0, \quad 6 \quad}$.

(3) Find an eigenvector for each eigenvalue.

Work (1 point):

Case 1: We compute the reduced row echelon form of the matrix $\lambda I_3 - A$ for $\lambda = 0$.

$$\begin{bmatrix} 0 - 2 & -2 & -2 \\ -2 & 0 - 2 & -2 \\ -2 & -2 & 0 - 2 \end{bmatrix} \longmapsto \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding equation is $x_1 + x_2 + x_3 = 0$, hence we find solutions $x_1 = -r - s, x_2 = r, x_3 = s$.

Case 2: We compute the reduced row echelon form of the matrix $\lambda I_3 - A$ for $\lambda = 6$.

$$\begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \mapsto \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding equation has solutions $x_1 = t, x_2 = t, x_3 = t$.
Answer (1 point):

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

You can verify that these vectors are indeed eigenvectors.

Problem 4 (10 points). *Consider the following* 4×5 *matrix* A *and a linear transformation* $L : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ *it defines:*

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 0 & 2 & 8 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

(1) What is the rank of the matrix A? (Work and answer count 1 point each.) ____________________.

As usual, we find the reduced row echelon form of A.

$$A \mapsto \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & 2 & 3 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & 2 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mapsto \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 9/5 & 0 \\ 0 & 0 & 1 & -1/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus the rank of A is 4.

(2) Find a basis for the kernel of L, and a basis for the range of L. (Work counts 3 points, and the answer counts 5 points.)

The solutions for the equation $A\mathbf{x} = \mathbf{0}$ are given by

$$x_1 = -r, x_2 = -\frac{9}{5}r, x_3 = \frac{1}{5}r, x_4 = r, x_5 = 0.$$

Thus we can take $\left\{ \begin{bmatrix} -5 \\ -9 \\ 1 \\ 5 \\ 0 \end{bmatrix} \right\}$ as a basis for $\mathrm{Ker}(L)$. The range of L is the span of the column vectors of A, and a basis is usually found by choosing the columns of A at the position of the corner 1's of the reduced row echelon form. Thus we obtain $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 8 \\ 1 \end{bmatrix} \right\}$ as a basis for the range of L.

But for this particular problem, since the dimension of the range ($= \mathrm{rank}(A)$) is 4 and hence L is onto, you can choose any basis for $\mathbb{R}^4$ as an answer.

Problem 5 (10 points). *Answer each of the following questions.*

(1) Let A be a 5×5 matrix whose eigenvalues are $-2, -1, 0, 1, 2$. Find the characteristic polynomial of A. $\underline{(\lambda+2)(\lambda+1)\lambda(\lambda-1)(\lambda-2)}$.

If you want to expand it, then

$$(\lambda+2)(\lambda+1)\lambda(\lambda-1)(\lambda-2) = \lambda(\lambda^2-4)(\lambda^2-1) = \lambda^5 - 5\lambda^3 + 4\lambda.$$

(2) Is the above matrix A diagonalizable? **Yes** **No**.

Since A has five distinct eigenvalues, it is diagonalizable.

(3) Find all the eigenvalues of A^3: $\underline{-8, -1, 0, 1, 8}$.

If $P^{-1}AP$ is a diagonal matrix, then

$$(P^{-1}AP)^3 = P^{-1}APP^{-1}APP^{-1}AP = P^{-1}A^3P.$$

Therefore, A^3 is also diagonalizable, and its eigenvalues are the cubes of the eigenvalues of A.

Problem 6 (5 points). *Let A be the following 3×3 matrix. Answer each of the questions below. (Circle the right answer if it is yes or no question.)*

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix}.$$

(1) Is the matrix A non-singular? **Yes**

(2) Is the matrix A an orthogonal matrix? **Yes**

(3) Is the matrix A diagonalizable? **Yes** (because it is symmetric.)

(4) What is the dimension of the kernel of A?

$$\dim \operatorname{Ker}(A) = \underline{\ 0\ }.$$

Since A is non-singular, the equation $A\mathbf{x} = \mathbf{0}$ has only one solution $\mathbf{x} = \mathbf{0}$. Therefore, $\operatorname{Ker}(A) = \{\mathbf{0}\}$.

(5) Find the inverse of A: $A^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$.

Since A is symmetric and orthogonal, we know $A = A^T = A^{-1}$.

Problem 7 (2 points)**.** *Let* $\mathbf{w}_1, \dots, \mathbf{w}_n$ *be vectors in* $\mathbb{R}^n$ *that satisfy the condition* $\mathbf{w}_i \cdot \mathbf{w}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$. *Show that* $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ *is linearly independent.*

Suppose $\sum_{i=1}^n c_i \mathbf{w}_i = \mathbf{0}$ for some real numbers $c_1, \dots, c_n$. Then for every index j, we have

$$0 = \mathbf{0} \cdot \mathbf{w}_j = \sum_{i=1}^n c_i \mathbf{w}_i \cdot \mathbf{w}_j = c_j \mathbf{w}_j \cdot \mathbf{w}_j = c_j.$$

Therefore, $c_1 = \cdots = c_n = 0$. Hence $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is linearly independent.

Problem 8 (1 point)**.** *Let* A *and* B *be* $n \times n$ *non-singular matrices. Show that* AB^{-1} *and* $B^{-1}A$ *are similar.*

Note that AB^{-1} is similar to $B^{-1}(AB^{-1})B$. But since $B^{-1}(AB^{-1})B = B^{-1}A$, AB^{-1} is similar to $B^{-1}A$.

Problem 9 (2 points)**.** *Let* A *be a symmetric matrix of size* $n \times n$. *Show that if* $\mathbf{u}$ *and* $\mathbf{v}$ *are two eigenvectors of* A *that belong to distinct eigenvalues, then they are mutually orthogonal.*

This one was done in class, and also is on the book. Please utilize your sources!