

MAT 22A: Linear Algebra Second Midterm Examination

Problem 1 (5 points). Consider the vector subspace W of \mathbb{R}^4 with basis

$$\left\{ \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 3 \\ -1 \end{bmatrix} \right\}.$$

Use the Gram-Schmidt orthogonalization procedure to find an orthonormal basis for W .

Work (2 point):

Let $\mathbf{u}_1 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 5 \\ 1 \\ 3 \\ -1 \end{bmatrix}$. Then $\|\mathbf{u}_1\| = \sqrt{4 \times 16} = 8$. As the first vector, we

choose $\mathbf{w}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. The second vector is calculated from $\mathbf{v}_2 = \mathbf{u}_2 - (\mathbf{u}_2 \cdot \mathbf{w}_1)\mathbf{w}_1 =$

$$\begin{bmatrix} 4 \\ 2 \\ 4 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}. \text{ Since } \|\mathbf{v}_2\| = 2, \text{ we have } \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}. \text{ Finally, the Gram-}$$

Schmidt process gives $\mathbf{v}_3 = \mathbf{u}_3 - (\mathbf{u}_3 \cdot \mathbf{w}_1)\mathbf{w}_1 - (\mathbf{u}_3 \cdot \mathbf{w}_2)\mathbf{w}_2 = \begin{bmatrix} 5 \\ 1 \\ 3 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$

This gives $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$

Answer (3 points): $\mathbf{w}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w}_3 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$

Problem 2 (5 points). Find the characteristic polynomial and eigenvalues of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Work (1 points):

$$\det \left(\lambda I_3 - \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right) = \det \begin{bmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{bmatrix} = (\lambda - 1)^3 - 1 - 1 - 3(\lambda - 1) \\ = \lambda^3 - 3\lambda^2 + 3\lambda - 3 - 3\lambda + 3 = \lambda^2(\lambda - 3).$$

Answer (4 points).

The characteristic polynomial = $\lambda^2(\lambda - 3)$.

The eigenvalues: $\lambda = \underline{0, 3}$.

Problem 3 (4 points). For each of the following statements, determine if it is true or false, and circle the correct answer.

(1) Any collection of $n + 1$ vectors in \mathbb{R}^n is linearly dependent.

True

(2) Let $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ be an orthonormal basis for \mathbb{R}^n , and \mathbf{v} a vector in the same vector space. Then the coordinate of \mathbf{v} with respect to this basis is $(\mathbf{w}_1 \cdot \mathbf{v}, \mathbf{w}_2 \cdot \mathbf{v}, \dots, \mathbf{w}_n \cdot \mathbf{v})^T$.

True

(3) An arbitrary straight line or an arbitrary plane is an example of a vector subspace of \mathbb{R}^3 .

False

(4) Let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be a basis for \mathbb{R}^n , where each vector is represented by a column vector. Define an $n \times n$ matrix $A = [\mathbf{u}_1 \ \mathbf{u}_2 \ \dots \ \mathbf{u}_n]$. Then $\det A = 0$.

False

Problem 4 (5 points). Consider the homogeneous system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

(1) Find a basis for the space of solutions of the equation.

Work (1 points): The reduced row echelon form of the matrix A is $\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus the solution of the equation $A\mathbf{x} = \mathbf{0}$ is $\begin{cases} x_1 = -r - s \\ x_2 = -r \\ x_3 = r \\ x_4 = r \\ x_5 = s \end{cases}$, with arbitrary pa-

rameters r and s . Therefore, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Answer (2 points): $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

(2) What is the dimension of the solution space? (1 point) 2.

(3) What is the rank of A ? (1 point) 3.

Problem 5 (3 points). *Let*

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

be two bases for \mathbb{R}^3 . Find the transition matrix $P_{S \leftarrow T}$ from the T -basis to the S -basis.

Remark. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Take an arbitrary vector \mathbf{v} of \mathbb{R}^3 . The S -coordinate of \mathbf{v} is calculated by finding an expression $\mathbf{v} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3$. Then

$[\mathbf{v}]_S = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Similarly, if $\mathbf{v} = y_1\mathbf{w}_1 + y_2\mathbf{w}_2 + y_3\mathbf{w}_3$, then $[\mathbf{v}]_T = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. The transition matrix

gives $[\mathbf{v}]_S = P_{S \leftarrow T}[\mathbf{v}]_T$, or equivalently, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = P_{S \leftarrow T} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. To find $P_{S \leftarrow T}$, we need to find

the S -coordinates of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. This means we have to solve $\mathbf{w}_i = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3$ for $i = 1, 2, 3$.

Work (1 point): Consider the augmented matrix $\begin{bmatrix} 1 & 1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 0 & 0 & 1 \end{bmatrix}$. The right-hand

side of the reduced row echelon form of this matrix is the transition matrix. We find that

the reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & -1 & 1 \\ 0 & 1 & 0 & \vdots & 0 & 1 & -1 \\ 0 & 0 & 1 & \vdots & 0 & 0 & 1 \end{bmatrix}$.

Answer (2 points): $P_{S \leftarrow T} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 6 (3 points). *Let* $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ *be an orthonormal basis for* \mathbb{R}^n , *where each vector is considered to be a column vector. Define an* $n \times n$ *matrix* $A = [\mathbf{u}_1 \ \mathbf{u}_2 \ \dots \ \mathbf{u}_n]$. *What is the determinant of* A ?

Answer (2 points): $\det A = \underline{\pm 1}$.

Reason (1 point): We note that $A^T A = I_n$. Therefore,

$$1 = \det I_n = \det(A^T A) = \det A^T \det A = (\det A)^2$$

since $\det A^T = \det A$. It follows that $\det A = \pm 1$.