MAT 21B: CALCULUS
THIRD MIDTERM EXAMINATION

DATE AND TIME: FRIDAY, MARCH 9, 2001. 9:00A.M.-9:50A.M.
ROOM: 2205 HARING
INSTRUCTOR: M. MULASE

Name: (Last) ____________________ (First) ____________________

Student ID Number: ______ – ______ – ______

CRN Number: ____________________

Remark.

1. The test consists of 5 pages, including the cover sheet.
2. **Do not de-staple the test.**
3. It is an open-book exam. Thus you can use anything written in the textbook and your lecture notes.
4. No calculators are allowed in the exam.
5. You can use scratch paper during the exam. However, you *cannot* submit any work on your scratch paper with the exam.

Scores:

Page 2: ____________/3

Page 3: ____________/6

Page 4: ____________/11

Page 5: ____________/8

Total: ____________/28

Problem 1 (9 points)**.** Let R be the infinite plane region bounded by the x-axis, the y-axis, and the graph of

$$y = \exp\left(-\frac{x}{2}\right).$$

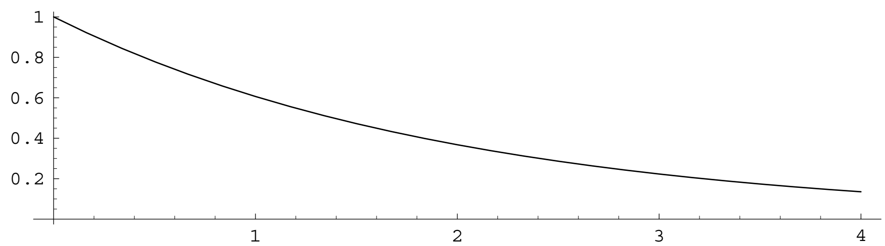

FIGURE 1. The plane region R under $y = e^{-x/2}$ for $x \geq 0$.

1. Find the x-coordinate of the centroid $(\overline{x}, \overline{y})$ of R.
   (a) Formula:

$$\overline{x} = \frac{\int_0^\infty xe^{-x/2}dx}{\int_0^\infty e^{-x/2}dx}.$$

   (b) Work: We have

$$\int_0^\infty e^{-x/2}dx = -2e^{-x/2}\Big|_0^\infty = 2,$$

   and

$$\int_0^\infty xe^{-x/2}dx = -2xe^{-x/2}\Big|_0^\infty - 4e^{-x/2}\Big|_0^\infty = 4.$$

   Thus $\overline{x} = 4/2 = 2$.

   (c) Choose the code of the correct answer from the list on Page 3:

$$\overline{x} = [\ p\ ].$$

2. Find the y-coordinate of the centroid $(\overline{x}, \overline{y})$ of R.
   (a) Formula:

$$\overline{y} = \frac{\int_0^\infty \frac{1}{2}(e^{-x/2})^2 dx}{\int_0^\infty e^{-x/2} dx}.$$

   (b) Work:

$$\int_0^\infty \frac{1}{2}(e^{-x/2})^2 dx = \int_0^\infty \frac{1}{2} e^{-x} dx$$
$$= -\frac{1}{2} e^{-x}\Big|_0^\infty = \frac{1}{2}.$$

   Thus $\overline{y} = 1/4$.

   (c) Choose the code of the correct answer from the list below:

$$\overline{y} = [\ m\ ].$$

3. Let V_x be the volume of the solid of revolution obtained by rotating R about the x-axis, V_y the volume of the solid obtained by rotating R about the y-axis, and V the volume of the solid obtained by rotating R about the line $x = -1$. Find the three volumes V_x, V_y and V.

   Choose the code of the correct answer from the list below:

$$V_x = [\ d\ ]; \qquad V_y = [\ h\ ]; \qquad V = [\ j\ ].$$

List: (a) $\frac{\pi}{8}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π (e) 2π (f) 4π
(g) 6π (h) 8π (i) 10π (j) 12π (k) 14π (l) 16π
(m) $\frac{1}{4}$ (n) $\frac{1}{2}$ (o) 1 (p) 2 (q) 4 (r) 6 (s) 8

Problem 2 (11 points)**.** Place the *most* appropriate word, quantity, or function in each [ ] of the following sentence.

The improper integral

$$\int_0^\infty \frac{\sin(x)}{x} dx$$

is [convergent]. To demonstrate the above statement, let us first decompose the integration into two parts:

$$(1) \qquad \int_0^\infty \frac{\sin(x)}{x} dx = \int_0^1 \frac{\sin(x)}{x} dx + \int_1^\infty \frac{\sin(x)}{x} dx.$$

Using the integration by parts, we have

$$(2) \qquad \int_1^\infty \frac{\sin(x)}{x} dx = \frac{[-\cos(x)]}{x}\Bigg|_1^\infty - \int_1^\infty \frac{[\cos(x)]}{[x^2]} dx.$$

Since the numerator of the first term of the right-hand side of Eq.(2) is bounded as $x \to \infty$, we have

$$\frac{[(\text{the same as above})]}{x}\Bigg|_1^\infty = [\cos(1)].$$

As to the second term of the right-hand side of Eq.(2), since $|\cos(x)| \leq 1$, we have

$$\int_1^\infty \left| \frac{[\cos(x)]}{[x^2]} \right| dx \leq \int_1^\infty \frac{1}{[\quad x^2 \quad]} dx = [\quad 1 \quad].$$

Therefore, the improper integral of the second term is absolutely convergent.

Finally, the integral

$$\int_0^1 \frac{\sin(x)}{x} dx$$

is convergent, because the function $\sin(x)/x$ is actually [continuous] at $x = 0$. Indeed, we have

$$\lim_{x \to 0} \frac{\sin(x)}{x} = [\quad 1 \quad].$$

Problem 3 (8 points)**.** Consider the curve $r = 1 + a\cos\theta$, known as a *limaçon*, where $a \geq 0$ is a constant (see Figure 2).

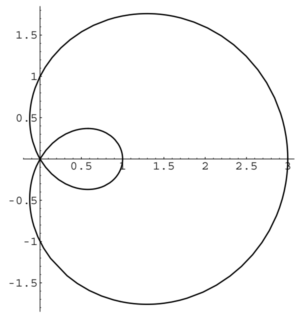

FIGURE 2. Limaçon for $a = 2$.

1. (1 point) Describe the curve for $a = 0$.

   Since the equation is $r = 1$, it is a circle of radius 1.

2. (2 points) Find the intersection of this limaçon and the x-axis, except for the origin.

   Setting $\theta = 0$, we have $r = 1 + a$. When $\theta = \pi$, $r = 1 - a$. Thus the intersection points are $(1 + a, 0)$ and $(-1 + a, 0)$ in the Cartesian coordinates.

3. (2 points) Find the intersection of this curve and the y-axis, except for the origin.

   When $\theta = \pi/2$, $r = 1$. When $\theta = -\pi/2$, we also have $r = 1$. Thus $(0, 1)$ and $(0, -1)$ are the intersection points in the Cartesian coordinates.

4. (3 points) You can trace the limaçon as θ changes from 0 to 2π. How many times does the curve pass through the origin? The answer depends on the value of a.

   We solve the equation $1 + a\cos(\theta) = 0$ for $0 \leq \theta < 2\pi$, or the equation $\cos(\theta) = -1/a$ for $a > 0$.
   Answer: If $0 \leq a < 1$, then 0 times. If $a = 1$, then only once. If $a > 1$, then twice.