

MAT 145: COMBINATORICS
FIRST MIDTERM EXAMINATION
SOLUTIONS

DATE AND TIME: FRIDAY, APRIL 25, 2003. 2:10 – 3:00
ROOM: 119 WELLMAN
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Problem 1 (12 points). *Fill each of the blank spaces with an appropriate word, phrase, value, or a sequence of mathematical symbols.*

The number of r -combinations of a given set T consisting of n elements is denoted by $\binom{n}{r}$. If $0 \leq r \leq n$ and r and n are integers, then its value is

$$(1) \quad \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

This number gives the number of subsets of cardinality r in T . The set T has a total of 2^n subsets. This can be proved as follows. Let $T = \{a_1, \dots, a_n\}$. To determine a subset $S \subset T$, we have to decide for each element a_j if $a_j \in S$ or not. Since there are 2 choices for each a_j , and since there are n elements in the set T , we have the desired result. The number of r -combinations is also called the *binomial coefficient*. This is because it appears in the binomial expansion formula

$$(2) \quad (1+x)^n = \sum_{r=0}^n \binom{n}{r} x^r.$$

Substituting $x = \underline{1}$, we obtain another proof for the number of subsets of T . If we differentiate Eqn.(2), we obtain

$$(3) \quad n(1+x)^{n-1} = n \cdot \sum_{r=0}^{n-1} \binom{n-1}{r} x^r$$

from the left-hand side, and differentiation of the RHS of Eqn.(2) gives

$$(4) \quad \sum_{r=0}^{n-1} (r+1) \binom{n}{r+1} x^r.$$

Comparing the coefficients of x^r in Eqn.(3) and Eqn.(4), we obtain a formula

$$(5) \quad n \binom{n-1}{r} = (r+1) \binom{n}{r+1}.$$

The above formula, Eqn.(5), also directly follows from the expression Eqn.(1). A combinatorial proof for Eqn.(5) goes as follows. We consider a group of n people, and wish to select a committee consisting of $r+1$ members, of whom one member serves as chair. The first way to count the number of possibilities is to designate a chairperson first and then select

other members, which gives the left-hand side of Eqn.(5). The second way is to select the committee first and then choose a chair among them.

Problem 2 (8 points). *Fill each of the blank spaces with an appropriate word, phrase, value, or a sequence of mathematical symbols.*

The binomial coefficients are generalized for non-integer values of n . For every real number a and a positive integer r , we define

$$(6) \quad \binom{a}{r} = \frac{a(a-1)(a-2)\cdots(a-r+1)}{r!}.$$

For $r = 0$, we simply define $\binom{a}{0} = 1$. For example, the $a = -1$ case has a simple expression $\binom{-1}{r} = \underline{(-1)^r}$. If $a = -n$ for a positive integer n , then $\binom{-n}{r}$ is the product of this quantity you have just derived and a single ordinary binomial coefficient of non-negative integers:

$$(7) \quad \binom{-n}{r} = \underline{(-1)^r} \times \binom{r+n-1}{r}.$$

Using the Taylor expansion theorem one can derive a similar formula like Eqn.(2), which is valid for $|x| < 1$:

$$(8) \quad (1-x)^{-n} = \sum_{r=0}^{\infty} \binom{r+n-1}{r} x^r.$$

Since $|x| < 1$, the LHS of Eqn.(8) can be computed as

$$(9) \quad (1-x)^{-n} = \left(\frac{1}{1-x}\right)^n = \overbrace{(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots)}^{n\text{-times}}.$$

If we choose x^{m_i} from the i -th factor in the RHS of Eqn.(9), then the coefficient of x^r in Eqn.(8) is equal to the number of non-negative integer solutions of the equation

$$(10) \quad m_1 + m_2 + \cdots + \underline{m_n} = \underline{r}.$$

There is a combinatorial way of directly counting the number of solutions of Eqn.(10). This is the number of ways of making a fruit basket containing a total of r fruits chosen from n different kinds of fruits, where we assume that there are infinitely many fruits available for each kind.

As a related problem, let us consider the case when you have 5 different kinds of fruits, and want to form a basket consisting of 12 fruits. If we require that the basket contains at least one fruit of each kind, then there are $\binom{11}{7}$ ways to form such a fruit basket. (No proof is required.)

Problem 3 (4 points). *Find a single binomial coefficient that is equal to the right hand side of the equation, where $0 \leq r \leq n$ are integers. (No proof is required.)*

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}, r \geq 1.$$

$$\binom{n+2}{r} = \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}, r \geq 2.$$

$$\binom{n+4}{r} = \binom{n}{r} + 4\binom{n}{r-1} + 6\binom{n}{r-2} + 4\binom{n}{r-3} + \binom{n}{r-4},$$

$$r \geq 4.$$

$$\binom{n+m}{r} = \sum_{p=0}^m \binom{m}{p} \binom{n}{r-p}, r \geq m.$$

Problem 4 (6 points). Let $n > 0$ be a positive integer. In how many ways can n boys and n girls be seated at a round table if the boys and girls are to sit in alternate seats, and we consider two seating the same if one is obtained by rotating the other?

Answer: $(n-1)!n!$ ways.

Reason: This is a question of circular permutation. Thus we can let one boy choose any seat of the table. Then all other boys can take the $n-1$ alternate seats. The number of choices for the $n-1$ boys is thus $(n-1)!$. After all boys have sit, then n girls take the n remaining seats. The number of such choices is $n!$. Therefore, the total number of distinct circular seating is $(n-1)!n!$.

Problem 5 (5 points). Consider a set $T = \{a_1, a_2, \dots, a_6\}$ consisting of 6 distinct integers ranging from 1 through 12. Show that there are two disjoint subsets S_1 and S_2 of T such that the sum of the elements of S_1 and that of S_2 are equal.

We notice that there are $2^6 = 64$ possible subsets in T . Thus the possible numbers of the sum of elements in a subset is at most 64. On the other hand, each subset has the sum of elements that can range from 0 to the maximum of $7+8+9+10+11+12 = 57$. (Note that T consists of *distinct* integers!) Since $64 > 57 + 1$, we conclude, by the pigeonhole principle, that there are two subsets, say T_1 and T_2 of T , such that the sum of elements in T_1 is equal to that of T_2 . If these subsets are disjoint, then we have established the proof. So suppose T_1 and T_2 have intersection. Then define

$$\begin{cases} S_1 = T_1 \setminus (T_1 \cap T_2) \\ S_2 = T_2 \setminus (T_1 \cap T_2) \end{cases}.$$

The sums of elements of S_1 and S_2 are the same, and by construction they are disjoint. This completes the proof.

Have a nice weekend!