MAT 21B: CALCULUS
PRACTICE FIRST MIDTERM EXAMINATION

DATE AND TIME: FRIDAY, JANUARY 26, 2001. 9:00A.M.-9:50A.M.
ROOM: 2205 HARING
INSTRUCTOR: M. MULASE

Name: (Last) ______________________ (First) ______________________

Student ID Number: ______ – ______ – ______

Discussion Section Number: ______________________

Remark.

1. The exam set has 6 pages.
2. **Do not detach the set.**
3. It is an open-book exam. Thus you can use anything written in the book. However, it is strongly advised that you memorize important formulas and mathematical procedures. Otherwise, it will take too much time to finish the exam!

Scores:

Page 2: ____________/4

Page 3: ____________/2

Page 4: ____________/6

Page 5: ____________/2

Page 6: ____________/6

Total: ____________/20

Problem 1 (4 points)**.** This problem concerns the volume of a solid that is obtained by rotating a curve

$$y = \sqrt{x}, \qquad x \text{ in } [0, a]$$

about the x axis.

1. Express this volume as a definite integral.
2. Compute the volume.

Solution. 1. The solid looks like a potato cut in half. Let us chop it into potato chips along planes perpendicular to the x-axis. Note that the volume is the integral of the area of the cross section at x for every x in the interval $[0, a]$. Since the cross section is the circle of radius $\sqrt{x}$, we have

$$\text{volume } = \int_0^a \pi(\sqrt{x})^2 dx.$$

2. We evaluate the integral.

$$\begin{aligned}\int_0^a \pi(\sqrt{x})^2 dx &= \int_0^a \pi x dx \\ &= \pi \int_0^a x dx \\ &= \pi \frac{a^2}{2}.\end{aligned}$$

Problem 2 (2 points, See Chapter 5, Review Problems, #60)**.** Consider a function $y = f(x)$ that satisfies the following conditions:

1. $f(x) \geq 0$ for every $x \geq 0$;
2. $f(x) > 0$ for every $x > 0$; and
3. an equation
$$\int_0^x f(t)dt = [f(x)]^2.$$

(a) Find $f(0)$.
(b) Find $f(x)$ for $x > 0$.

Solution.

(a) Since
$$0 = \int_0^0 f(t)dt = [f(0)]^2,$$
we conclude that $f(0) = 0$.

(b) By differentiating the condition 3 above, we obtain
$$f(x) = 2f(x)f'(x).$$
Since $f(x) > 0$ for $x > 0$, we can cancel $f(x)$, and obtain
$$1 = 2f'(x), \qquad \text{or} \qquad f'(x) = \frac{1}{2}.$$
Therefore,
$$f(x) = \frac{1}{2}x + c$$
for some constant c. Since $f(0) = 0$, we conclude that
$$f(x) = \frac{1}{2}x.$$

Problem 3 (6 points)**.**

1. Evaluate
$$\sum_{i=1}^{10} i^2.$$
2. Evaluate
$$\sum_{i=1001}^{2000} i.$$
3. Evaluate
$$\int_0^\pi \sin(x)dx.$$

Solution.

1. You'd better remember the formula
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$
otherwise it would take too much time.
$$\sum_{i=1}^{10} i^2 = \frac{10(10+1)(20+1)}{6} = 10 \times 11 \times 21 \div 6 = 385.$$
2. Again you have to know the formula
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$
$$\sum_{i=1001}^{2000} i = \sum_{i=1}^{2000} i - \sum_{i=1}^{1000} i = \frac{2000(2000+1)}{2} - \frac{1000(1000+1)}{2}$$
$$= 2001000 - 500500 = 1500500.$$
3. We know that $\int \sin x dx = -\cos x + c$. Thus
$$\int_0^\pi \sin x dx = -\cos(\pi) - (-\cos(0)) = 2.$$
Please read p. 304.

Problem 4 (2 points)**.** Find a function $f(x)$ such that $f(x) > 0$ for all x and that

$$f(x) = 3\int_0^x f(t)dt + 1. \tag{0.1}$$

Solution. The problem is slightly different from 6.7, #33.

First, we note that

$$f(0) = 3\int_0^0 f(t)dt + 1 = 0 + 1 = 1.$$

Next, we differentiate the given formula (0.1).

$$f'(x) = 3f(x).$$

Since $f(x) > 0$, we have

$$\frac{f'(x)}{f(x)} = 3.$$

Using a formula of logarithm

$$\frac{d\ln f(x)}{dx} = \frac{f'(x)}{f(x)},$$

we now have

$$\ln f(x) = \int 3dx = 3x + c,$$

where constant c is to be determined. Taking the exponential, we have

$$f(x) = e^{3x} \cdot e^c.$$

Since $f(0) = 1$, we know $e^c = 1$, hence

$$f(x) = e^{3x}.$$

Problem 5 (6 points)**.**

1. Evaluate
$$\int_0^1 x^2 e^x dx.$$
2. Find the antiderivative
$$\int \frac{tdt}{\sqrt{1-t^2}}.$$
Hint: Use substitution $u = 1 - t^2$.
3. Evaluate
$$\int_{-1}^1 x^3 dx.$$

Solution.

1. Use integration by parts. See page 416 to find the antiderivative
$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + c.$$
Thus
$$\int_0^1 x^2 e^x dx = e - 2.$$
2. Since $du = -2tdt$, we have
$$\begin{aligned}\int \frac{tdt}{\sqrt{1-t^2}} &= -\int \frac{du}{2\sqrt{u}} \\ &= -\sqrt{u} + c \\ &= -\sqrt{1-t^2} + c.\end{aligned}$$
3. Since $x^3 = -(-x)^3$, we find
$$\int_{-1}^1 x^3 dx = 0.$$