MAT 21B: CALCULUS
THIRD PRACTICE EXAMINATION

DATE AND TIME: FRIDAY, MARCH 9, 2001. 9:00A.M.-9:50A.M.
ROOM: 2205 HARING
INSTRUCTOR: M. MULASE

Name: (Last) (First) ______________________________

Student ID Number: – – ______________________________

CRN Number: ______________________________

Remark.

1. The test contains of 4 pages, including the cover sheet.
2. **Do not de-staple the test.**
3. It is an open-book exam. Thus you can use anything written in the textbook and your lecture notes.
4. No calculators are allowed in the exam.
5. You can use scratch paper during the exam. However, you *cannot* submit any work on your scratch paper with the exam.

Scores:

Page 2: ____________/8

Page 3: ____________/6

Page 4: ____________/6

Total: ____________/20

Problem 1 (8 points)**.** Let R be the plane region bounded by $y = x^2$ and $y = 1$ (see Figure 1).

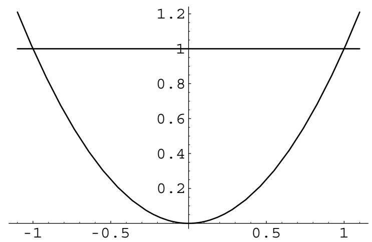

FIGURE 1. The plane region R.

1. Find the coordinate $(\overline{x}, \overline{y})$ of the centroid of R.

   Solution. We note that $\int_{-1}^{1}(1 - x^2)dx = 2 - \frac{2}{3} = \frac{4}{3}$ and $\int_{-1}^{1} \frac{1}{2}(1+x^2)(1-x^2)dx = 1 - \frac{1}{5} = \frac{4}{5}$. Thus $\overline{y} = \frac{4}{5} \cdot \frac{3}{4} = \frac{3}{5}$. $\overline{x} = 0$ because of the symmetry. Answer: $(0, 3/5)$.

2. Find the volume of the solid that is obtained by revolving R about the x-axis.

   Solution. The volume is $2\pi \cdot \frac{3}{5} \cdot \frac{4}{3} = \frac{8}{5}\pi$.

3. Find the volume of the solid that is obtained by revolving R about the line $x = 1$.

   Solution. The volume is $2\pi \cdot \frac{4}{3} = \frac{8}{3}\pi$.

4. Find the volume of the solid that is obtained by revolving R about the line $y = 1$.

   Solution. The volume is $2\pi \cdot \left(1 - \frac{3}{5}\right) \cdot \frac{4}{3} = \frac{16}{15}\pi$.

Problem 2 (6 points). 1. Evaluate

$$\int_0^\infty xe^{-x}dx.$$

Solution. By integration by parts, we find $\int xe^{-x}dx = -xe^{-x}+\int e^{-x}dx = -xe^{-x}-e^{-x}$. Recall that $\lim_{b\to\infty} be^{-b} = 0$. Thus we have

$$\int_0^\infty xe^{-x}dx = \lim_{b\to\infty}(-xe^{-x} - e^{-x})\big|_0^b = 1.$$

2. Evaluate

$$\int_0^\infty x^2e^{-x}dx.$$

Solution. Again by integration by parts, we have $\int x^2e^{-x}dx = -x^2e^{-x} + 2\int xe^{-x}dx$. Since $\lim_{b\to\infty} b^2e^{-b} = 0$, we have

$$\int_0^\infty x^2e^{-x}dx = \lim_{b\to\infty}(-x^2e^{-x})\big|_0^b + 2\int_0^\infty xe^{-x}dx = 2.$$

3. Evaluate

$$\int_0^\infty x^3e^{-x}dx.$$

Solution. We have

$$\int_0^\infty x^3e^{-x}dx = \lim_{b\to\infty}(-x^3e^{-x})\big|_0^b + 3\int_0^\infty x^2e^{-x}dx = 6.$$

Problem 3 (6 points)**.** Consider the curve $r = 1 + 2\cos\theta$, known as a *limaçon* (see Figure 2).

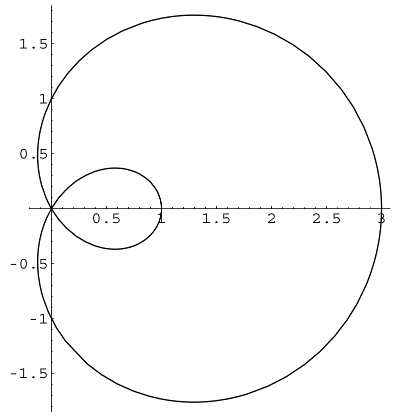

FIGURE 2. Limaçon.

1. Find the intersection of this limaçon and the x-axis.

   Solution. We examine $\theta = 0$, $\theta = \pi$, and $r = 0$. When $\theta = 0$, $r = 3$. For $\theta = \pi$, $r = -1$. This means the point is at $(1, 0)$ in Cartesian coordinate. Since r can be also 0, there are three x-intercepts. In Cartesian coordinate, they are $(3, 0)$, $(1, 0)$, and $(0, 0)$.

2. Find the intersection of this curve and the y-axis.

   Solution. We examine $\theta = \frac{\pi}{2}$, $\theta = \frac{3\pi}{2}$, and $r = 0$. In Cartesian coordinate, the y-intercepts are $(0, 1)$, $(0, -1)$, and $(0, 0)$.

3. You can trace the limaçon as θ changes from 0 to 2π. How many times does the curve pass through the origin?

   Solution. $r = 0 = 1 + 2\cos\theta$ implies $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. Thus the curve passes through the origin twice.