Character Table of the Symmetric Group, S4

isomorphic to the group of rotations of a cube

⇒ Using Division to reduce the problem.

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<u>Set of permutations</u> {1,2,3,4} with product operation of composition forms a group. Closure, associative, identity & inverses
Order of |S4| = (n)⋅(n-1)⋅(n-2)⋅1=4! = 24 elements. Multiplication table 24x24 matrix
Permutation types of {1,2,3,4} are:

four 1 cycles	(1)(2)(3)(4)	identity	1234→ 1234	$\equiv [1^4]$	⇒ $4!/(1^4*4!)$	= 1	g_1
one 2 cycle	(12)(3)(4)	transposition	1234→ 2134	$\equiv [1^2 2]$	⇒ $4!/(1^2*2*2!)$	= 6	g_2
one 3 cycle	(123)(4)	2 transpositions (12)(13)	1234→ 2314	$\equiv [13]$	⇒ 4!/(1*3)	= 8	g_3
one 4 cycle	(1234)	3 transpositions (12)(13)(14)	1234→ 2341	$\equiv [4]$	⇒ 4!/(4)	= 6	g_4
two 2 cycles	(12)(34)	2 transpositions	1234→ 2143	$\equiv [2^2]$	⇒ $4!(2^2*2!)$	= 3	g_5

<u>Conjugacy Class sizes</u> = # permutations each cycle type = $|S_4| / ((1^{\alpha_1}2^{\alpha_2}...n^{\alpha_n})(\alpha_1!\alpha_2!...\alpha_n!)) = |g_j|$
$(1^{\alpha_1}2^{\alpha_2}...n^{\alpha_n})$ are the same cycles, internal ordering (12)=(21); $(\alpha_1!\alpha_2!\ldots\alpha_n!)$ are the same cycles, external ordering (12)(34)=(34)(12).
5 Permutation Types = 5 Cycle Types ≡ 5 Conjugacy Classes = 5 Characters Equivalence Classes
⇒ Character Table of S_4 is a 5 x 5 matrix.

<u>Find the smallest Normal Subgroup to S4</u>
Lagrange – If G finite group & H subgroup of G, then |H| | |G|.
divisors of 24 are 1,2,3,4,6,8,12,24 |H| divides |G|, so |H| is equal to the divisors of |G|
Theorem – A subgroup N is normal to a group G, iff N is a union of the conjugacy classes of G, g_j.
Including the identity, the normal subgroups are:

$\{g_1\}$,	$\{g_1 \cup g_5\}$,	$\{g_1 \cup g_3 \cup g_5\}$,	$\{g_1 \cup g_2 \cup g_3 \cup g_4 \cup g_5\}$
1,	1+3=4,	1+3+8=12,	1+6+8+6+3=24
{1},	$\{C_2 \times C_2\}$,	$\{A_4\}$,	$\{S_4\}$

⇒ Smallest normal subgroup is $\{g_1 \cup g_5\}$ which is the conjugacy class of two 2cycles, $C_2 \times C_2$.

<u>Quotient Group (Factor Group)</u>
G/N = $S_4/C_2 \times C_2$ ⇒ $|S_4| / |C_2 \times C_2|$ = 24/(1+3) = 6 elements
= $\{h_1N, h_2N, h_3N, h_4N, h_5N, h_6N\}$ $g \in S_4$ left cosets
h_1N = N = {1, (12)(34), (13)(24), (14)(34)} Multiply the 4set, N, on the left by each of the 24 elements of G
h_2N = (123)N = {(123), (134), (142), (243)} to obtain the six left cosets of the quotient group $S_4/C_2 \times C_2$.
h_3N = (132)N = {(132), (124), (143), (234)}
h_4N = (12)N = {(12), (34), (1324), (1423)} All cosets have equal size, 4 elements.
h_5N = (13)N = {(13), (24), (1234), (1432)}
h_6N = (14)N = {(14), (23), (1243), (1342)}
For normal groups the cosets satisfy: $h_1Nh_2N = h_1h_2N$.
⇒ Allowing a 6x6 multiplication table to be built.

One than can see an <u>isomorphism between $S_4/C_2 \times C_2$ ≅ D_{2*3} ≅ S_3</u>.
D_{2*3} = {<a b> | $a^3=b^2=1$, $a^iba^i=b$} = {1, a, a^2, b, ba, ba^2}
= rigid motions of a triangle = {i, r, r^2, x, y, z}
S_3 = {1,(12),(13),(23),(123),(132)}

<u>Conjugacy Classes</u>	<u>g_1</u>	<u>g_2</u>	<u>g_3</u>
D_{2*3}	i	r	x
S_3	1	(12)	(123)
$S_4/C_2 \times C_2$	N	(123)N	(12)N

<u>Correspondence: D_{2*3}</u>	<u>S_3</u>	<u>$S_4/C_2 \times C_2$</u>
mapping sets to elements i	1	N
r	(123)	(123)N
r^2	(132)	(132)N
x	(12)	(12)N
y	(13)	(13)N
z	(14)	(14)N

Big Theorem (Lift Characters)

If N normal subgroup of G. We can create a bijective correspondence between the characters of G/N and G called the lift of the character, $\tilde{\chi}$ of G/N to the character, χ of G s. t. $\chi(g) = \tilde{\chi}(gN)$ $g \in G$.

if $k \in N$ then $\chi(k)=\tilde{\chi}(Nk)=\tilde{\chi}(N)=\chi(1)$ so $N \subseteq \text{Ker}\ \chi$.

Remark – Irreducible characters of S_3 correspond to irreducible characters of S_4.

Generate the Character Table of S_3

3 conjugacy classes, g_1=N, g_2= (12)N, g_3= (123)N

We always have the trivial character ⇒ $\chi_1(g_j) = 1$.

Than we have the sign character ⇒ $\chi_2(g_j) = 1$ if an even permutation and $\chi_2(g_j) = 0$ if an odd permutation.

And we have the character due to the permutation matrix ⇒ $\chi_3(g_j)$ = trace[permutation matrix] – $\chi_1(g_j)$.

g_1, ⇒ [100, 010, 001]. $\chi_3(g_1) = 3-1 = 2$.

g_2=(12)N ⇒ [010, 100, 001]. $\chi_3(g_2) = 1-1 = 0$.

g_3=(123)N ⇒ [010, 001, 100]. $\chi_3(g_3) = 0-1 = -1$.

$S_4/C_2\times C_2 \cong S_3$	N	(12)N	(123)N
χ_1 Trivial	1	1	1
χ_2 Sign	1	-1	1
χ_3 Permutation	2	0	-1

Lift the Characters of S_3 to S_4

We can copy/lift the first 3 conjugacy classes directly:

$\chi_j(1) = \chi_j(N)$, $\chi_j((12)) = \chi_j((12)N)$, $\chi_j((123)) = \chi_j((123)N)$.

The conjugacy class of two 2cycles is our normal group, $C_2\times C_2$: $\chi_j((12)(34)) = \chi_j(N)$.

The conjugacy class of a 4cycle is contained in h_4N, h_5N, and h_6N left cosets and therefore within the (12)N conjugacy class, so $\chi_j((1234)) = \chi_j((12)N)$.

S_4	1	(12)	(123)	(1234)	(12)(34)
χ_1 Trivial	1	1	1	1	1
χ_2 Sign	1	-1	1	-1	1
χ_3 Lifted from S_3	2	0	-1	0	2

Complete the Character Table of S_4

The permutation character $\chi_4(g_j)$ = trace[permutation matrix] – $\chi_1(g_j)$.

g_1, ⇒ [1000, 0100, 0010, 0001]. $\chi_4(g_1) = 4-1 = 3$.

g_2=(12)N ⇒ [0100, 1000, 0010, 0001]. $\chi_4(g_2) = 2-1 = 1$.

g_3=(123)N ⇒ [0100, 0010, 1000, 0001]. $\chi_4(g_3) = 1-1 = 0$.

g_4=(1234)N ⇒ [0100, 0010, 0001, 1000]. $\chi_4(g_4) = 0-1 = -1$.

g_5=(12)(34)N ⇒ [0100, 1000, 0001, 0010]. $\chi_4(g_5) = 0-1 = -1$.

We than multiply $\chi_2*\chi_4$ and get χ_5. Now we have completed the Character Table for S_4.

S_4	g_1 (1)	g_2 (12)	g_3 (123)	g_4 (1234)	g_5 (12)(34)
χ_1 Trivial	1	1	1	1	1
χ_2 Sign	1	-1	1	-1	1
χ_3 Lifted from S_3	2	0	-1	0	2
χ_4 Permutation	3	1	0	-1	-1
$\chi_5 = \chi_2*\chi_4$	3	-1	0	1	-1

Summary: This is an example of generating characters by lifting them from smaller normal quotient groups.

Extra Notes: S_4={a,b,c,d} are the diagonals formed from opposite vertices of a cube, a=17 b=28 c=35 d=46.
$C_2\times C_2$ is formed from the three 180° x,y,z cube rotations, swapping the diagonals: (ab)(cd), (ac)(bd), (ad)(bc).