POLYNOMIAL RECURSION FORMULA FOR LINEAR HODGE INTEGRALS

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Abstract. We establish a polynomial recursion formula for linear Hodge integrals. It is obtained as the cut-and-join equation for the Laplace transform of the Hurwitz numbers. We show that the recursion recovers the Witten-Kontsevich theorem when restricted to the top degree terms, and also the combinatorial factor of the λ_g formula as the lowest degree terms.

Dedicated to Herbert Kurke on the occasion of his 70th birthday

Contents

1. Introduction

The purpose of this paper is to establish a topological recursion formula for linear Hodge integrals in terms of *polynomial* generating functions. Let $\overline{\mathcal{M}}_{g,\ell}$ be the Deligne-Mumford moduli stack of stable curves of genus g and ℓ distinct marked points subject to $2g-2+\ell>0$. We denote by ψ_i the i-th cotangent class of $\overline{\mathcal{M}}_{g,\ell}$, and by $\lambda_j = c_j(\mathbb{E})$ the j-th Chern class of the Hodge bundle $\mathbb{E}$ on $\overline{\mathcal{M}}_{g,\ell}$. By *linear* Hodge integrals we mean the rational numbers

$$\langle \tau_{n_1}\cdots\tau_{n_\ell}\lambda_j\rangle_{g,\ell} = \int_{\overline{\mathcal{M}}_{g,\ell}} \psi_1^{n_1}\cdots\psi_\ell^{n_\ell}\lambda_j.$$

Following [7, 14] we define a series of polynomials by a recursion formula

$$\hat{\xi}_{n+1}(t) = t^2(t-1)\frac{d}{dt}\hat{\xi}_n(t) = D\hat{\xi}_n(t)$$

with the initial condition $\hat{\xi}_0(t) = t-1$. The differential operator $D = t^2(t-1)\frac{d}{dt}$ found in [14] simplifies many of the combinatorial difficulties of the linear Hodge integrals and Hurwitz numbers. The degree of $\hat{\xi}_n(t)$ is $2n+1$. We consider symmetric polynomials of degree $3(2g-2+\ell)$,

$$\widehat{\mathcal{H}}_{g,\ell}(t_1,\ldots,t_\ell) = \sum_{n_1,\ldots,n_\ell} \langle \tau_{n_1}\cdots\tau_{n_\ell}\Lambda_g^\vee(1)\rangle_{g,\ell}\prod_{i=1}^{\ell}\hat{\xi}_{n_i}(t_i), \tag{1.1}$$

where $\Lambda_g^\vee(1) = 1-\lambda_1+\cdots+(-1)^g\lambda_g$. The following is our main theorem.

2000 *Mathematics Subject Classification.* 14H10, 14N10, 14N35; 05A15, 05A17; 81T45.

Theorem 1.1. *The polynomial generating functions of the linear Hodge integrals* (1.1) *satisfy the following topological recursion formula*

$$(1.2)\quad \left(2g-2+\ell+\sum_{i=1}^{\ell}\frac{1}{t_i}D_i\right)\widehat{\mathcal{H}}_{g,\ell}(t_L)$$
$$=\sum_{i<j}\frac{t_i^2\hat{\xi}_0(t_j)D_i\widehat{\mathcal{H}}_{g,\ell-1}\left(t_{L\setminus\{j\}}\right)-t_j^2\hat{\xi}_0(t_i)D_j\widehat{\mathcal{H}}_{g,\ell-1}\left(t_{L\setminus\{i\}}\right)}{t_i-t_j}$$
$$+\sum_{i=1}^{\ell}\left[D_{u_1}D_{u_2}\widehat{\mathcal{H}}_{g-1,\ell+1}\left(u_1,u_2,t_{L\setminus\{i\}}\right)\right]_{u_1=u_2=t_i}$$
$$+\frac{1}{2}\sum_{i=1}^{\ell}\sum_{\substack{g_1+g_2=g\\ J\sqcup K=L\setminus\{i\}}}^{\text{stable}}D_i\widehat{\mathcal{H}}_{g_1,|J|+1}(t_i,t_J)\cdot D_i\widehat{\mathcal{H}}_{g_2,|K|+1}(t_i,t_K),$$

where $D_i=t_i^2(t_i-1)\frac{\partial}{\partial t_i}$. *The last summation is taken over all partitions* $g=g_1+g_2$ *of the genus* g *and disjoint union decompositions* $J\sqcup K=L\setminus\{i\}$ *satisfying the stability conditions* $2g_1-1+|J|>0$ *and* $2g_2-1+|K|>0$. *Here* $L=\{1,2,\ldots,\ell\}$ *is the index set, and for a subset* $I\subset L$ *we write* $t_I=(t_i)_{i\in I}$.

The recursion formula (1.2) is a *topological* recursion in the sense that it gives the generating function of linear Hodge integrals of complexity $2g-2+\ell=n$ in terms of those of complexity $n-1$. The same topological structure appears in other recursion formulas such as those discussed in [5, 8, 9, 14, 25, 26, 27, 28].

We prove Theorem 1.1 by computing the Laplace transform of the Hurwitz number $h_{g,\mu}$ as a *function* of a partition μ. Let $f:X\to\mathbb{P}^1$ be a morphism of connected nonsingular algebraic curve X of genus g onto the projective line defined over $\mathbb{C}$. If we regard f as a meromorphic function on X, then the *profile* of f is the list of orders of its poles being considered as a *partition* of the degree of f. The *Hurwitz* number $h_{g,\mu}$ we deal with in this paper is the number of topological types of f of given genus g and profile μ being counted with the weight $1/|\mathrm{Aut}(f)|$. The celebrated *cut-and-join equation* of Goulden, Jackson, and Vakil [12, 31] applied to the Laplace transformed Hurwitz numbers is exactly the polynomial recursion (1.2). The idea of taking the Laplace transform of the cut-and-join equation comes from [7]. It is shown in [7] that (1.2) implies the Bouchard-Mariño conjecture on the topological recursion for Hurwitz numbers [3], which is the simplest case of the more general conjecture on the closed and open Gromov-Witten invariants of toric Calabi-Yau 3-folds [2].

The significance of (1.2) being a polynomial is two-fold. Firstly, the leading coefficients of $\widehat{\mathcal{H}}_{g,\ell}$ are the ψ-class intersection numbers. It was proved by Okounkov and Pandharipande [29] that the large partition asymptotics of the Hurwitz numbers recover the Witten-Kontsevich theorem, i.e., the Virasoro constraint condition of the ψ-class intersection numbers [5, 21, 32]. Since the Laplace transform contains more information than the asymptotic behavior, the proof becomes just to compare the leading coefficients of the polynomial equation (1.2). The second significance is that the coefficients of the *lowest* degree terms are the linear Hodge integrals containing the λ_g-class. The topological recursion restricts to the recursion formula for linear Hodge integrals $\langle\tau_{n_1}\cdots\tau_{n_\ell}\lambda_g\rangle_{g,\ell}$ with respect to ℓ. We also remark that the same *polynomiality* is observed in [18, 19] in the context of integrable systems.

We note that all the formulas in this paper have been established in various different formulations [4, 13, 14, 15, 18, 22]. Since (1.2) is equivalent to the cut-and-join equation, logically there is nothing new. The contribution of this paper is the simple expression of our formulation of the cut-and-join equation (3.14) and a new point of view of understanding (1.2) as the Laplace transform of (3.14). It gives a clear and unified picture of some of the results established in [4, 15, 18].

The paper is organized as follows. We begin with setting our notations and reviewing definitions of Hurwitz numbers in Section 2. In Section 3 we formulate the cut-and-join equation as a functional equation for functions in partitions. Although there are a large number of literature on the subject [4, 12, 13, 14, 15, 18, 19, 22, 24, 31, 33], we provide a full detail in this section because we wish to arrive at a simpler formulation of the equation. We then introduce the idea of Laplace transformation following [7] in Section 4. Here the role of the Lambert curve, the *spectral curve* of the topological recursion for Hurwitz numbers introduced in [1, 3, 7, 8], is identified as the *Riemann surface* of a meromorphic function that is obtained by the Laplace transform. The following Section 5 establishes Theorem 1.1. In the final section we derive the Dijkgraaf-Verlinde-Verlinde formula [5] for the Witten-Kontsevich theory [21, 32] from (1.2) as its simple corollary. We also study the λ_g formula [10, 11] from the point of view of topological recursion.

Acknowledgement. The authors thank the American Institute of Mathematics for the hospitality during their stay that promoted this collaboration, and Lin Chen for useful comments. M.M. thanks Herbert Kurke for giving him the opportunity to lecture on Hurwitz numbers based on [29, 30] at the Humboldt University of Berlin in 2002 and 2005. M.M. also thanks the Institute for the Physics and Mathematics of the Universe, the Osaka City University Advanced Mathematical Institute, the NSF, Kyoto University, Tôhoku University, KIAS in Seoul, and the University of Salamanca for their hospitality and financial support during the preparation of this work.

2. Hurwitz numbers

Let X be a nonsingular complete algebraic curve of genus g defined over the complex number field $\mathbb{C}$, and $f : X \to \mathbb{P}^1$ a morphism of X to the projective line $\mathbb{P}^1$. If we regard f a meromorphic function on the Riemann surface X, then the inverse image $f^{-1}(\infty) = \{p_1, \dots, p_\ell\}$ of $\infty \in \mathbb{P}^1$ is the set of poles of f. We can name these ℓ points so that the list of pole orders becomes a *partition* $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell > 0)$ of the degree of the map. Thus the *size* of this partition $|\mu| = \mu_1 + \cdots + \mu_\ell$ is $\deg f$, and its *length* $\ell(\mu) = \ell$ is the number of poles of f. Each part μ_i determines a local description of the map f, which is given by $z \longmapsto z^{\mu_i}$ in terms of a local coordinate z of X around p_i. A critical point, or a *ramification point*, of f is a point $p \in X$ at which the derivative vanishes $df(p) = 0$, and $w = f(p)$ is a critical value, or a *branched point* of f. Let $B \subset \mathbb{P}^1$ be the set of all branched points of f. Then

$$f\big|_{f^{-1}(\mathbb{P}^1 \setminus B)} : f^{-1}(\mathbb{P}^1 \setminus B) \longrightarrow \mathbb{P}^1 \setminus B \tag{2.1}$$

is a topological covering of degree $|\mu|$. When the derivative df has a simple zero at p, we say p is a *simple ramification point* of f. If over every branched point except for ∞ there is exactly one simple ramification point, then we call f a *Hurwitz cover*. The partition μ gives the *profile* of a Hurwitz cover. The number $h_{g,\mu}$ of topological types of Hurwitz covers of given genus g and profile μ, counted with the weight factor $1/|\text{Aut} f|$, is the *Hurwitz number* we are interested in this paper. To be more precise, we study $h_{g,\mu}$ as a *function* of partition μ. We will compute the *Laplace transform* of $h_{g,\mu}$ and find the equations that they satisfy.

Let r denote the number of simple ramification points of f. This gives the dimension of the Hurwitz scheme, i.e., the moduli space of all Hurwitz covers for a given genus and a profile [29]. Since (2.1) is a topological covering, the Euler characteristic of $f^{-1}(\mathbb{P}^1 \setminus B)$ is given by

$$\chi\big(f^{-1}(\mathbb{P}^1 \setminus B)\big) = \deg f \cdot \chi(\mathbb{P}^1 \setminus B) = |\mu|(1-r).$$

On the other hand, since $f^{-1}(x)$ contains exactly $\deg f - 1$ points for every $x \in B \setminus \{\infty\}$ and since $f^{-1}(\infty)$ has ℓ points,

$$\chi\big(f^{-1}(\mathbb{P}^1 \setminus B)\big) = 2 - 2g(X) - \ell - r(|\mu|-1).$$

We thus obtain the *Riemann-Hurwitz formula*

$$r = r(g,\mu) = 2g - 2 + \ell + |\mu|. \tag{2.2}$$

The celebrated Ekedahl-Lando-Shapiro-Vainshtein formula [6, 16, 29] relates Hurwitz numbers and linear Hodge integrals on the Deligne-Mumford moduli stack $\overline{\mathcal{M}}_{g,\ell}$ consisting of stable algebraic curves of genus g with ℓ distinct nonsingular marked points subject to the stability condition $2g-2+\ell > 0$. Denote by $\pi_{g,\ell} : \overline{\mathcal{M}}_{g,\ell+1} \to \overline{\mathcal{M}}_{g,\ell}$ the natural projection and by $\omega_{\pi_{g,\ell}}$ the relative dualizing sheaf of the universal curve $\pi_{g,\ell}$. The *Hodge* bundle $\mathbb{E}$ on $\overline{\mathcal{M}}_{g,\ell}$ is defined by $\mathbb{E} = (\pi_{g,\ell})_* \omega_{\pi_{g,\ell}}$, and the λ-classes are the Chern classes

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,\ell}, \mathbb{Q})$$

of the Hodge bundle. Let $\sigma_i : \overline{\mathcal{M}}_{g,\ell} \to \overline{\mathcal{M}}_{g,\ell+1}$ be the i-th tautological section of π, and put $\mathcal{L}_i = \sigma_i^*(\omega_{\pi_{g,\ell}})$. The ψ-classes are defined by

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,\ell}, \mathbb{Q}).$$

The *linear Hodge integrals* are rational numbers defined by

$$\langle \tau_{n_1} \cdots \tau_{n_\ell} \lambda_j \rangle_{g,\ell} = \int_{\overline{\mathcal{M}}_{g,\ell}} \psi_1^{n_1} \cdots \psi_\ell^{n_\ell} \lambda_j,$$

which are 0 unless $n_1 + \cdots + n_\ell + j = 3g - 3 + \ell$. Let us denote by $\Lambda_g^\vee(1) = 1 - \lambda_1 + \cdots + (-1)^g \lambda_g$. The ELSV formula states

$$h_{g,\mu} = \frac{r(g,\mu)!}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{\ell(\mu)} \big(1 - \mu_i \psi_i\big)}, \tag{2.3}$$

where $\mathrm{Aut}(\mu)$ is the permutation group that interchanges the equal parts of μ. The appearance of this automorphism factor is due to the difference between giving a profile μ and *naming* all points in $f^{-1}(\infty)$. If all parts of μ are distinct, then the poles of f are naturally labeled by the pole order. But when two or more parts are the same, there is no way to distinguish the Hurwitz covers obtained by interchanging these poles of the same order. The factor $1/|\mathrm{Aut}(\mu)|$ takes care of this overount.

Although $\overline{\mathcal{M}}_{g,\ell}$ is defined as the moduli stack of *stable* curves satisfying the stability condition $2-2g-\ell < 0$, Hurwitz numbers are well defined for *unstable* geometries $(g,\ell) = (0,1)$ and $(0,2)$. It is an elementary exercise to show that

$$h_{0,k} = k^{k-3} \qquad \text{and} \qquad h_{0,(\mu_1,\mu_2)} = \frac{(\mu_1+\mu_2)!}{\mu_1+\mu_2} \cdot \frac{\mu_1^{\mu_1}}{\mu_1!} \cdot \frac{\mu_2^{\mu_2}}{\mu_2!}.$$

The ELSV formula remains true for unstable cases by *defining*

$$\int_{\overline{\mathcal{M}}_{0,1}} \frac{\Lambda_0^\vee(1)}{1 - k\psi} = \frac{1}{k^2}, \tag{2.4}$$

$$\int_{\overline{\mathcal{M}}_{0,2}} \frac{\Lambda_0^\vee(1)}{(1-\mu_1\psi_1)(1-\mu_2\psi_2)} = \frac{1}{\mu_1+\mu_2}. \tag{2.5}$$

3. The cut-and-join equation

The Hurwitz numbers satisfy a set of combinatorial equations called the *cut-and-join* equation discovered in [12, 31]. It has become an effective tool of algebraic geometry for studying Hurwitz numbers and many related subjects [4, 13, 14, 15, 16, 17, 19, 24, 29, 33]. In this section we review the equation following [12, 22, 31, 33], and give its simplest formulation that is suitable to compute its Laplace transform in Section 5.

The topological covering (2.1) gives rise to a unique point in the *character variety*

$$\rho \in \mathrm{Hom}\big(\pi_1(\mathbb{P}^1 \setminus B), S_d\big)\big/S_d, \tag{3.1}$$

where S_d is the symmetric group of $d = |\mu|$ letters and its action on the set of homomorphisms is through conjugation. Since the character variety classifies *all* topological coverings, we need to determine the condition for a covering to be a Hurwitz cover. Let us list the $r+1$ points in B as

$$B = \{x_1, \dots, x_r, \infty\}.$$

Choose a base point $*$ on $\mathbb{P}^1 \setminus B$, and denote by γ_k a closed path starting from $*$ that goes around x_k in the positive direction, and comes back to $*$. The loop γ_∞ is the loop going around ∞. Then up to conjugation, we have

$$\pi_1(\mathbb{P}^1 \setminus B) \cong \langle \gamma_1, \dots, \gamma_r, \gamma_\infty \mid \gamma_1 \cdots \gamma_r \cdot \gamma_\infty = 1\rangle.$$

Now recall that over each x_k there is only one ramification point, say p_k, which is simple. Therefore, in terms of the representation ρ corresponding to the Hurwitz cover f, the generator γ_k is mapped to a transposition $(ab) \in S_d$. Next, recall that the ramification behavior over ∞ is determined by the profile μ, and that each part μ_i determines the map f locally as $z \longmapsto z^{\mu_i}$. In terms of the representation, this means that

$$\rho(\gamma_\infty) = c_1 c_2 \cdots c_\ell,$$

where

$$c_1 \sqcup \cdots \sqcup c_\ell = \{1, 2, \dots, d\}$$

is a disjoint cycle decomposition of the index set and each c_i is a cycle of length μ_i.

The cut-and-join equation represents the number of Hurwitz covers of a given genus g and profile μ in terms of those with profiles obtained by either *cutting* a part into two pieces, or *joining* two parts together. Let $p \in X$ be a point at which the covering $f : X \to \mathbb{P}^1$ is simply ramified. Locally we can name sheets, so we assume sheets a and b are ramified over $x_r = f(p) \in B \subset \mathbb{P}^1$. In terms of the representation we have $\rho(\gamma_r) = (ab) \in S_d$. When we merge x_r to ∞, the generators γ_r and γ_∞ of $\pi_1(\mathbb{P}^1 \setminus B)$ are replaces by their product $\gamma_r\gamma_\infty$. The representation ρ maps this generator to $(ab)c_1 \cdots c_\ell$. Now one of the two things happen:

(1) The *cut case*, in which both sheets are ramified at the same point p_i of the inverse image $f^{-1}(\infty) = \{p_1, \dots, p_\ell\}$. In terms of ρ, this means both indices a and b are contained in the same cycle c_i. Since $c_1, \dots, c_\ell$ are disjoint, we only need to calculate $(ab)c_i$. By re-naming all the sheets and assuming $a < b = a + \alpha < \mu_i = \alpha + \beta$, we can compute

$$\begin{aligned}&\big(a[a+\alpha]\big)\big(12\cdots[a-1]a[a+1]\cdots[a+\alpha]\cdots[\alpha+\beta]\big)\\ &= \big(a[a+1]\cdots[a+\alpha-1]\big)\big([a+\alpha][a+\alpha+1]\cdots[\alpha+\beta]12\cdots[a-1]\big).\end{aligned}$$

The result is the product of two disjoint cycles of length α and β. Thus the merging eliminate a profile μ and creates a new profile

$$(\mu_1, \dots, \widehat{\mu_i}, \dots, \mu_\ell, \alpha, \beta) = \big(\mu(\hat{i}), \alpha, \beta\big)$$

of length $\ell + 1$. Here the $\widehat{\ }$ sign means removing the entry. Note that the size of the partition $|\mu|$ is unchanged, because it is the degree of the map f. When α is chosen, the total number of such cuttings is $\alpha + \beta$ because this is the number of choices for a in the index set $\{1, 2, \dots, \alpha + \beta\}$. We also note that when $\alpha = \beta$, the number is actually α, instead of $\alpha + \beta$.

(2) The *join case*, in which sheets a and b are ramified at two distinct points, say p_i and p_j, above ∞. In other words, $a \in c_i$ and $b \in c_j$. Again by re-numbering, we can calculate

$$\begin{aligned}&(ab)\big(12\cdots[a-1]a[a+1]\cdots\mu_i\big)\big([\mu_i+1]\cdots[b-1]b[b+1]\cdots[\mu_i+\mu_j]\big)\\ &= \big(12\cdots[a-1]b[b+1]\cdots[\mu_i+\mu_j][\mu_j+1]\cdots[b-1]a[a+1]\cdots\mu_i\big).\end{aligned}$$

Thus the result of merging creates a new profile

$$(\mu_1, \dots, \widehat{\mu_i}, \dots, \widehat{\mu_j}, \dots, \mu_\ell, \mu_i + \mu_j) = \big(\mu(\hat{i}, \hat{j}), \mu_i + \mu_j\big)$$

of length $\ell - 1$ and size $|\mu|$. The total number of ways to make the join is $\mu_i \mu_j$, because we have μ_i-choices for a and μ_j-choices for b.

To utilize the above consideration into Hurwitz numbers, let us introduce the generating function of Hurwitz numbers

$$\mathbf{H}(s, \mathbf{p}) = \sum_{g\geq 0}\sum_{\ell\geq 1} \mathbf{H}_{g,\ell}(s, \mathbf{p}); \qquad \mathbf{H}_{g,\ell}(s, \mathbf{p}) = \sum_{\mu:\ell(\mu)=\ell} h_{g,\mu}\mathbf{p}_\mu \frac{s^{r(g,\mu)}}{r(g,\mu)!}, \tag{3.2}$$

where $\mathbf{p}_\mu = p_{\mu_1}p_{\mu_2}\cdots p_{\mu_\ell}$, and $r(g, \mu)$ is the number of simple ramification points (2.2). The summation in $\mathbf{H}_{g,\ell}(s, \mathbf{p})$ is over all partitions of length ℓ. Here $p_1, p_2, p_3, \dots$ are parameters that encode the information of partitions. The other parameter s counts the number r of simple ramification points. Since r and μ recover the genus g, s is a *topological* parameter. Note that merging x_r to ∞ means decreasing r by 1, or differentiating the generating function with respect to s. The result of this differentiation is the cut and join operations discussed above. Here we need to note that the cut cases may cause a disconnected covering of $\mathbb{P}^1$. Recall that the *exponential* generating function

$$e^{\mathbf{H}(s,\mathbf{p})} = 1 + \mathbf{H}(s, \mathbf{p}) + \frac{1}{2}\,\mathbf{H}(s, \mathbf{p})^2 + \frac{1}{3!}\,\mathbf{H}(s, \mathbf{p})^3 + \cdots$$

counts disconnected Hurwitz coverings. The power of $\mathbf{H}(s, \mathbf{p})$ is the number of connected components. Now the above merging consideration gives the following equation, which is the cut-and-join equation as a linear partial differential equation

$$\left[\frac{\partial}{\partial s} - \frac{1}{2}\sum_{\alpha,\beta\geq 1}\left((\alpha+\beta)p_\alpha p_\beta\frac{\partial}{\partial p_{\alpha+\beta}} + \alpha\beta p_{\alpha+\beta}\frac{\partial^2}{\partial p_\alpha \partial p_\beta}\right)\right] e^{\mathbf{H}(s,\mathbf{p})} = 0. \tag{3.3}$$

We can immediately deduce

$$\frac{\partial \mathbf{H}}{\partial s} = \frac{1}{2}\sum_{\alpha,\beta\geq 1}\left((\alpha+\beta)p_\alpha p_\beta\frac{\partial \mathbf{H}}{\partial p_{\alpha+\beta}} + \alpha\beta p_{\alpha+\beta}\frac{\partial^2 \mathbf{H}}{\partial p_\alpha \partial p_\beta} + \alpha\beta p_{\alpha+\beta}\frac{\partial \mathbf{H}}{\partial p_\alpha}\cdot\frac{\partial \mathbf{H}}{\partial p_\beta}\right). \tag{3.4}$$

This is the cut-and-join equation for the generating function $\mathbf{H}(s, \mathbf{p})$ of the number of *connected* Hurwitz coverings.

At this stage, we apply the ELSV formula (2.3) to (3.2). For a partition μ of length ℓ, we define

$$H_g(\mu) = \frac{|\mathrm{Aut}(\mu)|}{r(g,\mu)!} \cdot h_{g,\mu} = \sum_{n_1+\cdots+n_\ell \le 3g-3+\ell} \langle \tau_{n_1} \cdots \tau_{n_\ell} \Lambda_g^\vee(1) \rangle \prod_{i=1}^{\ell} \frac{\mu_i^{\mu_i+n_i}}{\mu_i!}. \tag{3.5}$$

Then we have

$$\mathbf{H}_{g,\ell}(s,\mathbf{p}) = \sum_{\mu:\ell(\mu)=\ell} \frac{1}{|\mathrm{Aut}(\mu)|} H_g(\mu)\mathbf{p}_\mu s^{r(g,\mu)} = \frac{1}{\ell!} \sum_{(\mu_1,\dots,\mu_\ell)\in\mathbb{N}^\ell} H_g(\mu)\mathbf{p}_\mu s^{r(g,\mu)}. \tag{3.6}$$

The automorphism factor $|\mathrm{Aut}(\mu)|$ in the formula comes from the re-summation. For any function $f(\mu)$ in μ, we have a change of summation formula

$$\sum_{\mu\in\mathbb{N}^\ell} f(\mu) = \sum_{\mu:\ell(\mu)=\ell} \frac{1}{\big|\mathrm{Aut}(\mu)\big|} \sum_{\sigma\in S_\ell} f(\mu^\sigma), \tag{3.7}$$

where S_ℓ is the permutation group of ℓ letters and

$$\mu^\sigma = \big(\mu_{\sigma(1)}, \dots, \mu_{\sigma(\ell)}\big) \in \mathbb{N}^\ell$$

is the integer vector obtained by permuting the parts of μ by $\sigma \in S_\ell$. If $f(\mu)$ is a symmetric function, then the summation over S_ℓ simply contributes $\ell!$ to the formula, as in (3.6). For a partition μ, let us denote by $m_\alpha(\mu)$ the multiplicity of α in μ, i.e., the number of α repeated in μ. Then we have

$$\big|\mathrm{Aut}(\mu)\big| = \prod_{k\ge1} m_k(\mu)!. \tag{3.8}$$

Let us now compare the coefficient of $\mathbf{p}_\mu s^{r-1}$ in the cut-and-join equation (3.4) for a given partition μ and an integer $r \ge 1$. The LHS contributes

$$r(g,\mu)\, \frac{H_g(\mu)}{|\mathrm{Aut}(\mu)|}, \tag{3.9}$$

subject to the condition $r = r(g,\mu)$.

The terms of $\mathbf{p}_\mu s^{r-1}$ that come from the *cut*-operation of the RHS of (3.4) must have a profile $\big(\mu(\hat{i},\hat{j}), \mu_i+\mu_j\big)$, because

$$\begin{aligned} r\Big(g, \big(\mu(\hat{i},\hat{j}), \mu_i+\mu_j\big)\Big) &= 2g-2+\ell\big(\mu(\hat{i},\hat{j}), \mu_i+\mu_j\big) + \Big|\big(\mu(\hat{i},\hat{j}), \mu_i+\mu_j\big)\Big| \\ &= 2g-2+(\ell-1)+|\mu| = r(g,\mu)-1. \end{aligned}$$

We see that the application of the differential operator $p_{\mu_i}p_{\mu_j}\partial/\partial p_{\mu_i+\mu_j}$ to $\mathbf{H}(s,\mathbf{p})$ restores the profile μ from $\big(\mu(\hat{i},\hat{j}), \mu_i+\mu_j\big)$. Thus the coefficient of $\mathbf{p}_\mu s^{r-1}$ is

$$\frac{1}{\big|\mathrm{Aut}(\mu)\big|} \sum_{i<j} (\mu_i+\mu_j) H_g\big(\mu(\hat{i},\hat{j}), \mu_i+\mu_j\big). \tag{3.10}$$

In this consideration, we are naming all parts of μ to apply the cut-operation. Therefore, we need to compensate the overcount by the $\mathrm{Aut}(\mu)$-factor. In terms of combinatorics, we can obtain (3.10) in a different way. Recall that [33] we have

$$\Big|\mathrm{Aut}\big(\mu(\hat{i},\hat{j}), \mu_i+\mu_j\big)\Big| = \begin{cases} \big|\mathrm{Aut}(\mu)\big| \cdot \frac{m_{\mu_i+\mu_j}(\mu)+1}{m_{\mu_i}(\mu)m_{\mu_j}(\mu)} & \mu_i \ne \mu_j, \\ \big|\mathrm{Aut}(\mu)\big| \cdot \frac{m_{\mu_i+\mu_j}(\mu)+1}{m_{\mu_i}(\mu)\big(m_{\mu_i}(\mu)-1\big)} & \mu_i = \mu_j. \end{cases} \tag{3.11}$$

So if $\mu_i \neq \mu_j$, then

$$\frac{1}{|\mathrm{Aut}(\mu)|}(\mu_i+\mu_j)H_g\big(\mu(\hat{i},\hat{j}),\mu_i+\mu_j\big)$$

$$= (\mu_i+\mu_j)\cdot\frac{m_{\mu_i+\mu_j}(\mu)+1}{m_{\mu_i}(\mu)m_{\mu_j}(\mu)}\cdot\frac{H_g\big(\mu(\hat{i},\hat{j}),\mu_i+\mu_j\big)}{\Big|\mathrm{Aut}\big(\mu(\hat{i},\hat{j}),\mu_i+\mu_j\big)\Big|},$$

where each factor of the RHS has combinatorial significance. When $\mu_i=\mu_j=\alpha$, we have

$$\frac{1}{|\mathrm{Aut}(\mu)|}(\mu_i+\mu_j)H_g\big(\mu(\hat{i},\hat{j}),\mu_i+\mu_j\big) = \alpha\cdot\frac{m_{2\alpha}(\mu)+1}{\binom{m_\alpha(\mu)}{2}}\cdot\frac{H_g\big(\mu(\hat{i},\hat{j}),2\alpha\big)}{\Big|\mathrm{Aut}\big(\mu(\hat{i},\hat{j}),2\alpha\big)\Big|},$$

where the part α is removed from the i-th and j-th slots of μ.

In a *join* term we must have a profile $\big(\mu(\hat{i}),\alpha,\beta\big)$. Since $\ell\big(\mu(\hat{i}),\alpha,\beta\big)=\ell+1$, changing r to $r-1$ requires reducing the genus. One possibility is

$$\begin{aligned} r\Big(g-1,\big(\mu(\hat{i}),\alpha,\beta\big)\Big) &= 2(g-1)-2+\ell\big(\mu(\hat{i}),\alpha,\beta\big)+\Big|\big(\mu(\hat{i}),\alpha,\beta\big)\Big| \\ &= 2g-2+(\ell+1)+|\mu|-2 = r(g,\mu)-1. \end{aligned}$$

In this case the differential operator $p_{\alpha+\beta}\partial^2/\partial p_\alpha\partial p_\beta$ applied to $\mathbf{H}(s,\mathbf{p})$ recovers the profile μ. The coefficient of $\mathbf{p}_\mu s^{r-1}$ is then

$$\frac{1}{2|\mathrm{Aut}(\mu)|}\sum_{i=1}^{\ell}\sum_{\alpha+\beta=\mu_i}\alpha\beta H_{g-1}\big(\mu(\hat{i}),\alpha,\beta\big). \tag{3.12}$$

Here again we can give a combinatorial explanation of this formula using (3.8) and (3.11). When $\alpha\neq\beta$, we have

$$\frac{1}{|\mathrm{Aut}(\mu)|}\alpha\beta H_{g-1}\big(\mu(\hat{i}),\alpha,\beta\big) = \alpha\beta\cdot\frac{\big(m_\alpha(\mu)+1\big)\big(m_\beta(\mu)+1\big)}{m_{\mu_i}(\mu)}\cdot\frac{H_{g-1}\big(\mu(\hat{i}),\alpha,\beta\big)}{\Big|\mathrm{Aut}\big(\mu(\hat{i}),\alpha,\beta\big)\Big|}.$$

And if $\alpha=\beta=\frac{1}{2}\mu_i$, then

$$\frac{1}{|\mathrm{Aut}(\mu)|}\alpha^2 H_{g-1}\big(\mu(\hat{i}),\alpha,\alpha\big) = 2\alpha^2\cdot\frac{\binom{m_\alpha(\mu)+2}{2}}{m_{\mu_i}(\mu)}\cdot\frac{H_{g-1}\big(\mu(\hat{i}),\alpha,\alpha\big)}{\Big|\mathrm{Aut}\big(\mu(\hat{i}),\alpha,\alpha\big)\Big|}.$$

The overall factor 2 in the RHS comes from the second order differentiation $\partial^2/\partial p_\alpha^2$.

There is yet another possibility to obtain the profile μ from a *join*-operation, if we utilize disconnected Hurwitz covers. Consider Hurwitz covers

$$f_1 : X_1 \longrightarrow \mathbb{P}^1 \qquad \text{and} \qquad f_2 : X_2 \longrightarrow \mathbb{P}^1$$

of genus g_1 (*resp.* g_2) and profile (ν_1,α) (*resp.* (ν_2,β)). Let $\nu_1\sqcup\nu_2$ denote the partition obtained by gathering all parts of ν_1 and ν_2 together. If $g_1+g_2=g$ and $\nu_1\sqcup\nu_2=\mu(\hat{i})$, then the join-operation recovers the profile μ, provided that $\alpha+\beta=\mu_i$. This is because

$$\begin{aligned} r\big(g_1,(\nu_1,\alpha)\big) &= 2g_1-2+\ell(\nu_1)+1+|\nu_1|+\alpha \\ r\big(g_2,(\nu_2,\beta)\big) &= 2g_2-2+\ell(\nu_2)+1+|\nu_2|+\beta \\ r(g,\mu)-1 &= 2g-2+\ell+|\mu|-1. \end{aligned}$$

The $\mathbf{p}_\mu s^{r-1}$-term comes from $p_{\alpha+\beta}\frac{\partial \mathbf{H}}{\partial p_\alpha}\cdot\frac{\partial \mathbf{H}}{\partial p_\beta}$, and its coefficient is

$$\frac{1}{2\big|\mathrm{Aut}(\mu)\big|}\sum_{i=1}^{\ell}\sum_{\alpha+\beta=\mu_i}\alpha\beta\sum_{\substack{g_1+g_2=g\\ \nu_1\sqcup\nu_2=\mu(\hat{i})}}H_{g_1}(\nu_1,\alpha)H_{g_2}(\nu_2,\beta). \tag{3.13}$$

The combinatorial derivation of this formula follows from the identity

$$|\mathrm{Aut}(\nu_1\sqcup\nu_2)| = \big|\mathrm{Aut}(\nu_1)\big|\cdot\big|\mathrm{Aut}(\nu_2)\big|\cdot\prod_{k\geq 1}\binom{m_k(\nu_1\sqcup\nu_2)}{m_k(\nu_1)}.$$

When $\alpha\neq\beta$, we have

$$\begin{aligned}&\frac{1}{\big|\mathrm{Aut}(\mu)\big|}\alpha\beta\sum_{\substack{g_1+g_2=g\\ \nu_1\sqcup\nu_2=\mu(\hat{i})}}H_{g_1}(\nu_1,\alpha)H_{g_2}(\nu_2,\beta)\\ &\quad=\alpha\beta\cdot\frac{\big(m_\alpha(\mu)+1\big)\big(m_\beta(\mu)+1\big)}{m_{\mu_i}(\mu)}\cdot\frac{1}{\prod_{k\geq 1}\binom{m_k(\mu(\hat{i}),\alpha,\beta)}{m_k(\nu_1,\alpha)}}\cdot\frac{H_{g_1}(\nu_1,\alpha)}{\big|\mathrm{Aut}(\nu_1,\alpha)\big|}\cdot\frac{H_{g_2}(\nu_2,\beta)}{\big|\mathrm{Aut}(\nu_2,\beta)\big|}.\end{aligned}$$

And if $\alpha=\beta=\frac{1}{2}\mu_i$, then

$$\begin{aligned}&\frac{1}{\big|\mathrm{Aut}(\mu)\big|}\alpha^2\sum_{\substack{g_1+g_2=g\\ \nu_1\sqcup\nu_2=\mu(\hat{i})}}H_{g_1}(\nu_1,\alpha)H_{g_2}(\nu_2,\alpha)\\ &\quad=2\alpha^2\cdot\frac{\binom{m_\alpha(\mu)+2}{2}}{m_{\mu_i}(\mu)}\cdot\frac{1}{\prod_{k\geq 1}\binom{m_k(\mu(\hat{i}),\alpha,\alpha)}{m_k(\nu_1,\alpha)}}\cdot\frac{H_{g_1}(\nu_1,\alpha)}{\big|\mathrm{Aut}(\nu_1,\alpha)\big|}\cdot\frac{H_{g_2}(\nu_2,\alpha)}{\big|\mathrm{Aut}(\nu_2,\alpha)\big|}.\end{aligned}$$

Assembling (3.10), (3.12), and (3.13) together, we obtain the combinatorial form of the cut-and-join equation.

Theorem 3.1 (Cut-and-join equation)**.** *The functions $H_g(\mu)$ of* (3.5) *satisfy a recursion equation*

$$\begin{aligned}r(g,\mu)H_g(\mu) &= \sum_{i<j}(\mu_i+\mu_j)H_g\big(\mu(\hat{i},\hat{j}),\mu_i+\mu_j\big)\\ &+\frac{1}{2}\sum_{i=1}^{\ell}\sum_{\alpha+\beta=\mu_i}\alpha\beta\left(H_{g-1}\big(\mu(\hat{i}),\alpha,\beta\big)+\sum_{\substack{g_1+g_2=g\\ \nu_1\sqcup\nu_2=\mu(\hat{i})}}H_{g_1}(\nu_1,\alpha)H_{g_2}(\nu_2,\beta)\right).\end{aligned} \tag{3.14}$$

4. Laplace transform and the Lambert curve

Since linear Hodge integrals $\langle\tau_{n_1}\cdots\tau_{n_\ell}\lambda_j\rangle$ do not depend on a partition μ, it is natural to ask if there is any direct recursion formula for them *without* any reference to partitions. The answer is *yes*, and we give the formula in the next Section 5. The natural complexity measure for the moduli space $\overline{\mathcal{M}}_{g,\ell}$ is the absolute value $2g-2+\ell$ of the Euler characteristic of an ℓ-punctured Riemann surface of genus g. An inductive formula associated to $\overline{\mathcal{M}}_{g,\ell}$ with respect to $2g-2+\ell$ is generally called a *topological recursion.* We wish to establish a topological recursion for linear Hodge integrals. In the light of (3.5) and the combinatorial cut-and-join equation (3.14), it is obvious what we should do to eliminate the μ-dependence:

just take the summation over all partitions μ. This is the idea of the *Laplace transform* discovered in [7]. In this section we explain this idea.

Since the sum of $\frac{k^{k+n}}{k!}$ for all positive integer k diverges, we are naturally led to the idea of Laplace transformation. Indeed,

$$f_n(w) = \sum_{k=1}^{\infty} \frac{k^{k+n}}{k!} e^{-k(w+1)} \tag{4.1}$$

is a holomorphic function in w for $Re(w) > 0$. This follows from Stirling's formula

$$e^{-k} \frac{k^{k+n}}{k!} \sim \frac{1}{\sqrt{2\pi}}\, k^{n-\frac{1}{2}} \qquad \text{for} \qquad k >> 1.$$

Since

$$\int_0^{\infty} x^{n-\frac{1}{2}} e^{-xw} dx = \frac{\Gamma(n+\frac{1}{2})}{w^{n+\frac{1}{2}}} \tag{4.2}$$

for $n > -\frac{1}{2}$, $f_n(w)$ of (4.1) is expected to be a function of $\sqrt{w}$, instead of w itself, if n is an integer. We now come to the point of asking: *what is the Riemann surface of the function* $f_n(w)$? If the estimate (4.2) is exact, then the Riemann surface of $f_n(w)$ is the same as that of $\sqrt{w}$. But since it is not, we need a different idea.

The idea used in [7] is the following. First we introduce a function

$$t = t(w) = 1 + \sum_{k=1}^{\infty} \frac{k^k}{k!} e^{-k(w+1)}, \tag{4.3}$$

which is holomorphic for $Re(w) > 0$, and define

$$x = e^{-(w+1)} \qquad \text{and} \qquad y = \frac{t-1}{t}. \tag{4.4}$$

We can solve $t = t(w)$ in terms of x and y. The result is

$$x = ye^{-y}. \tag{4.5}$$

Let us call the plane analytic curve

$$C = \{(x,y) \in \mathbb{C}^2 \mid x = ye^{-y}\} \subset \mathbb{C}^2 \tag{4.6}$$

the *Lambert curve*. This naming is due to the resemblance of (4.3) and the classical *Lambert W-function*

$$W(x) = -\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (-x)^k.$$

The Lambert curve C is analytically isomorphic to $\mathbb{C}$, so it is an open Riemann surface of genus 0. The x-projection $\pi : C \to \mathbb{C}$ has a unique critical point $q_0 = (e^{-1}, 1) \in C$. In terms of the coordinates w and t, the inverse function of (4.3), or the equation for the Lambert curve, is given by

$$w = w(t) = -\frac{1}{t} - \log\left(1 - \frac{1}{t}\right) = \sum_{m=2}^{\infty} \frac{1}{m} \frac{1}{t^m}, \tag{4.7}$$

which is holomorphic for $Re(t) > 1$. The critical point of the projection π in this coordinate is $(w,t) = (0,\infty)$. Since the infinite series of (4.7) starts at $m = 2$, π is locally a double-sheeted covering around $w = 0$. And this is what we wanted. Indeed, the Lambert curve C

is *the Riemann surface* of the function $f_n(w)$. It is natural to consider $f_n(w)$ as a function in t, since t is a global coordinate of C. So we re-define

$$\hat{\xi}_n(t) = \sum_{k=1}^{\infty} \frac{k^{k+n}}{k!} e^{-k(w+1)}, \tag{4.8}$$

which is simply $f_n(w)$ in terms of t satisfying $w = w(t)$. But something amazing happens here: $\hat{\xi}_n(t)$ is a *polynomial* in t if $n \geq 0$. The proof is trivial. A standard property of the Laplace transform gives

$$-\frac{d}{dw} f_n(w) = \sum_{k=1}^{\infty} \frac{k^{k+n+1}}{k!} e^{-k(w+1)} = f_{n+1}(w), \tag{4.9}$$

and the coordinate change (4.7) implies

$$-\frac{d}{dw} = t^2(t-1)\frac{d}{dt}. \tag{4.10}$$

Therefore, $\hat{\xi}_n(t)$'s satisfy a recursion formula

$$\hat{\xi}_{n+1}(t) = t^2(t-1)\frac{d}{dt}\hat{\xi}_n(t) = D\hat{\xi}_n(t). \tag{4.11}$$

Since $\hat{\xi}_0(t) = t-1$ from (4.3), we see that $\hat{\xi}_n(t)$ is a polynomial in t of degree $2n+1$. It immediately follows that the Laplace transform

$$\begin{aligned} \widehat{\mathcal{H}}_{g,\ell}(t_1, \dots, t_\ell) &= \sum_{\mu \in \mathbb{N}^\ell} H_g(\mu) e^{-(\mu_1(w_1+1)+\cdots+\mu_\ell(w_\ell+1))} \\ &= \sum_{n_1+\cdots+n_\ell \leq 3g-3+\ell} \langle \tau_{n_1} \cdots \tau_{n_\ell} \Lambda_g^\vee(1) \rangle \prod_{i=1}^{\ell} \hat{\xi}_{n_i}(t_i) \end{aligned} \tag{4.12}$$

of $H_g(\mu)$ of (3.5) is a symmetric *polynomial* in the t-variables and naturally lives on C^ℓ, when $2g-2+\ell > 0$.

The unstable geometries $(g,\ell) = (0,1)$ and $(0,2)$ are the exceptions of this general formula. Recall the $(0,1)$ case (2.4). We have

$$\widehat{\mathcal{H}}_{0,1}(t) = \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} e^{-k(w+1)} = -\frac{1}{2\,t^2} + c = \hat{\xi}_{-2}(t), \tag{4.13}$$

where the constant c is given by

$$c = \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} e^{-k}.$$

The $(0,2)$ case (2.5) is quite more involved. It is proved in [7] that we have

$$\begin{aligned} \widehat{\mathcal{H}}_{0,2}(t_1, t_2) &= \sum_{\mu_1, \mu_2 \geq 1} \frac{1}{\mu_1 + \mu_2} \cdot \frac{\mu_1^{\mu_1}}{\mu_1!} \cdot \frac{\mu_2^{\mu_2}}{\mu_2!} e^{-\mu_1(w_1+1)} e^{-\mu_2(w_2+1)} \\ &= \log\left(\frac{\hat{\xi}_{-1}(t_1) - \hat{\xi}_{-1}(t_2)}{x_1 - x_2}\right) - \hat{\xi}_{-1}(t_1) - \hat{\xi}_{-1}(t_2), \end{aligned} \tag{4.14}$$

where

$$\hat{\xi}_{-1}(t) = \frac{t-1}{t} = y. \tag{4.15}$$

5. The topological recursion as a Laplace transform

In the previous section we have computed the Laplace transform of $H_g(\mu)$ as a function on partitions μ. In this section we calculate the Laplace transform of the cut-and-join equation (3.14) and proves Theorem 1.1.

Let us denote

$$\langle \mu, w+1\rangle = \mu_1(w_1+1)+\cdots+\mu_\ell(w_\ell+1). \tag{5.1}$$

Recalling the expression of $r(g,\mu)$ given in (2.2) and using (3.7), the Laplace transform of the LHS of (3.14) becomes

$$\sum_{\mu\in\mathbb{N}^\ell} r(g,\mu)H_g(\mu)e^{-\langle\mu,w+1\rangle} = \left(2g-2+\ell+\sum_{i=1}^{\ell} t_i^2(t_i-1)\frac{\partial}{\partial t_i}\right)\widehat{\mathcal{H}}_{g,\ell}(t_1,\dots,t_\ell). \tag{5.2}$$

Here we note that multiplication of μ_i to the summand corresponds to the operation of $D_i = t_i^2(t_i-1)\frac{\partial}{\partial t_i}$ due to (4.10).

To find the Laplace transform of the cut terms (3.10), we first note a formula:

$$\begin{aligned}\sum_{\mu_1,\mu_2\ge 0} f(\mu_1+\mu_2)e^{-(\mu_1 w_1+\mu_2 w_2)} &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} f(k)e^{-kw_1}e^{-m(w_2-w_1)}\\ &= \sum_{k=0}^{\infty}\frac{1-e^{-(k+1)(w_2-w_1)}}{1-e^{-(w_2-w_1)}}\, f(k)e^{-kw_1}\\ &= \frac{1}{e^{-w_1}-e^{-w_2}}\sum_{k=0}^{\infty} f(k)\left(e^{-(k+1)w_1}-e^{-(k+1)w_2}\right).\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\frac{1}{2}\sum_{\mu\in\mathbb{N}^\ell}\sum_{i\ne j}(\mu_i+\mu_j)H_g\left(\mu(\hat{i},\hat{j}),\mu_i+\mu_j\right)e^{-\langle\mu,w+1\rangle}\\ &= \frac{1}{2}\sum_{i\ne j}\frac{1}{e^{-(w_i+1)}-e^{-(w_j+1)}}\Bigg(e^{-(w_i+1)}t_i^2(t_i-1)\frac{\partial}{\partial t_i}\widehat{\mathcal{H}}_{g,\ell-1}\left(t_1,\dots,\widehat{t_j},\dots,t_\ell\right)\\ &\qquad - e^{-(w_j+1)}t_j^2(t_j-1)\frac{\partial}{\partial t_j}\widehat{\mathcal{H}}_{g,\ell-1}\left(t_1,\dots,\widehat{t_i},\dots,t_\ell\right)\Bigg)\\ &\qquad - \sum_{i\ne j} t_i^2(t_i-1)\frac{\partial}{\partial t_i}\widehat{\mathcal{H}}_{g,\ell-1}\left(t_1,\dots,\widehat{t_j},\dots,t_\ell\right),\end{aligned}$$

where the last term comes from the adjustment of the cases $\mu_i=0$ and $\mu_j=0$ that are not included in the Laplace transform.

The Laplace transform of the first join terms (3.12) is given by

$$\begin{aligned}&\frac{1}{2}\sum_{\mu\in\mathbb{N}^\ell}\sum_{i=1}^{\ell}\sum_{\alpha+\beta=\mu_i}\alpha\beta H_{g-1}\big(\mu(\hat{i}),\alpha,\beta\big)e^{-\langle\mu,w+1\rangle}\\ &\qquad = \sum_{i=1}^{\ell}\left[u_1^2(u_1-1)u_2^2(u_2-1)\frac{\partial^2}{\partial u_1\partial u_2}\widehat{\mathcal{H}}_{g-1,\ell+1}\left(u_1,u_2,t_{L\setminus\{i\}}\right)\right]_{u_1=u_2=t_i},\end{aligned}$$

where $t_I = (t_i)_{i\in I}$ for a subset $I \subset L = \{1, 2, \dots, \ell\}$. In the same way we can calculate the Laplace transform of the second join terms (3.13):

$$\sum_{\mu\in\mathbb{N}^\ell}\sum_{\alpha+\beta=\mu_i}\alpha\beta\sum_{\substack{g_1+g_2=g\\ \nu_1\sqcup\nu_2=\mu(\hat{i})}} H_{g_1}(\nu_1,\alpha)H_{g_2}(\nu_2,\beta)e^{-\langle\mu,w+1\rangle}$$
$$=\left[\sum_{\substack{g_1+g_2=g\\ J\sqcup K=L\setminus\{i\}}} u_1^2(u_1-1)\frac{\partial}{\partial u_1}\widehat{\mathcal{H}}_{g_1,|J|+1}(u_1,t_J)u_2^2(u_2-1)\frac{\partial}{\partial u_2}\widehat{\mathcal{H}}_{g_2,|K|+1}(u_2,t_K)\right]_{u_1=u_2=t_i}.$$

Thus we establish

$$\left(2g-2+\ell+\sum_{i=1}^\ell t_i^2(t_i-1)\frac{\partial}{\partial t_i}\right)\widehat{\mathcal{H}}_{g,\ell}(t_1,\dots,t_\ell) \tag{5.3}$$
$$=\sum_{i<j}\frac{1}{e^{-(w_i+1)}-e^{-(w_j+1)}}\Bigg(e^{-(w_i+1)}t_i^2(t_i-1)\frac{\partial}{\partial t_i}\widehat{\mathcal{H}}_{g,\ell-1}\left(t_1,\dots,\widehat{t_j},\dots,t_\ell\right)$$
$$-e^{-(w_j+1)}t_j^2(t_j-1)\frac{\partial}{\partial t_j}\widehat{\mathcal{H}}_{g,\ell-1}\left(t_1,\dots,\widehat{t_i},\dots,t_\ell\right)\Bigg)$$
$$-\sum_{i\neq j}t_i^2(t_i-1)\frac{\partial}{\partial t_i}\widehat{\mathcal{H}}_{g,\ell-1}\left(t_1,\dots,\widehat{t_j},\dots,t_\ell\right)$$
$$+\sum_{i=1}^\ell\left[u_1^2(u_1-1)u_2^2(u_2-1)\frac{\partial^2}{\partial u_1\partial u_2}\widehat{\mathcal{H}}_{g-1,\ell+1}\left(u_1,u_2,t_{L\setminus\{i\}}\right)\right]_{u_1=u_2=t_i}$$
$$+\frac{1}{2}\sum_{i=1}^\ell\sum_{\substack{g_1+g_2=g\\ J\sqcup K=L\setminus\{i\}}} t_i^2(t_i-1)\frac{\partial}{\partial t_i}\widehat{\mathcal{H}}_{g_1,|J|+1}(t_i,t_J)\cdot t_i^2(t_i-1)\frac{\partial}{\partial t_i}\widehat{\mathcal{H}}_{g_2,|K|+1}(t_i,t_K).$$

Note that *unstable geometries* are contained in the last summation. We use (4.13) and (4.14) to substitute the values in (5.3). The result becomes surprisingly simple due to cancellation of the non-polynomial terms. For $g_1 = 0$ and $J = \emptyset$, the contribution is

$$\sum_{i=1}^\ell \hat{\xi}_{-1}(t_i)t_i^2(t_i-1)\frac{\partial}{\partial t_i}\widehat{\mathcal{H}}_{g,\ell}(t_1,\dots,t_\ell).$$

For $g_1 = 0$ and $J = \{j\} \subset L\setminus\{i\}$, we have

$$t_i^2(t_i-1)\frac{\partial}{\partial t_i}\widehat{\mathcal{H}}_{0,2}(t_i,t_j)=\frac{\hat{\xi}_0(t_i)}{\hat{\xi}_{-1}(t_i)-\hat{\xi}_{-1}(t_j)}-\frac{x_i}{x_i-x_j}-\hat{\xi}_0(t_i)$$
$$=\frac{\hat{\xi}_0(t_i)}{\hat{\xi}_{-1}(t_i)-\hat{\xi}_{-1}(t_j)}-\frac{e^{-(w_i+1)}}{e^{-(w_i+1)}-e^{-(w_j+1)}}-\hat{\xi}_0(t_i).$$

Thus the unstable $(0, 2)$ contribution in (5.3) is

$$\sum_{i<j}\frac{t_i^2(t_i-1)^2\frac{\partial}{\partial t_i}\widehat{\mathcal{H}}_{g,\ell-1}\left(t_1,\dots,\widehat{t_j},\dots,t_\ell\right)-t_j^2(t_j-1)^2\frac{\partial}{\partial t_j}\widehat{\mathcal{H}}_{g,\ell-1}\left(t_1,\dots,\widehat{t_i},\dots,t_\ell\right)}{\hat{\xi}_{-1}(t_i)-\hat{\xi}_{-1}(t_j)}$$

$$- \sum_{i<j} \frac{1}{e^{-(w_i+1)} - e^{-(w_j+1)}} \Bigg(e^{-(w_i+1)} t_i^2 (t_i - 1) \frac{\partial}{\partial t_i} \widehat{\mathcal{H}}_{g,\ell-1} \left(t_1, \dots, \widehat{t_j}, \dots, t_\ell\right)$$

$$- e^{-(w_j+1)} t_j^2 (t_j - 1) \frac{\partial}{\partial t_j} \widehat{\mathcal{H}}_{g,\ell-1} \left(t_1, \dots, \widehat{t_i}, \dots, t_\ell\right) \Bigg)$$

$$- \sum_{i \neq j} \hat{\xi}_0(t_i) t_i^2 (t_i - 1) \frac{\partial}{\partial t_i} \widehat{\mathcal{H}}_{g,\ell-1} \left(t_1, \dots, \widehat{t_j}, \dots, t_\ell\right).$$

We have thus proved the following, which is equivalent to Theorem 1.1.

Theorem 5.1. *The Laplace transform of the cut-and-join equation is the following equation for polynomials $\widehat{\mathcal{H}}_{g,\ell}(t_1, \dots, t_\ell)$ subject to the stability condition $2g - 2 + \ell > 0$:*

$$\left(2g - 2 + \ell + \sum_{i=1}^{\ell} \left(1 - \hat{\xi}_{-1}(t_i)\right) t_i^2 (t_i - 1) \frac{\partial}{\partial t_i}\right) \widehat{\mathcal{H}}_{g,\ell}(t_1, \dots, t_\ell) \tag{5.4}$$

$$= \sum_{i<j} t_i t_j \frac{t_i^2 (t_i - 1)^2 \frac{\partial}{\partial t_i} \widehat{\mathcal{H}}_{g,\ell-1} \left(t_1, \dots, \widehat{t_j}, \dots, t_\ell\right) - t_j^2 (t_j - 1)^2 \frac{\partial}{\partial t_j} \widehat{\mathcal{H}}_{g,\ell-1} \left(t_1, \dots, \widehat{t_i}, \dots, t_\ell\right)}{t_i - t_j}$$

$$- \sum_{i \neq j} t_i^3 (t_i - 1) \frac{\partial}{\partial t_i} \widehat{\mathcal{H}}_{g,\ell-1} \left(t_1, \dots, \widehat{t_j}, \dots, t_\ell\right)$$

$$+ \frac{1}{2} \sum_{i=1}^{\ell} \left[u_1^2 (u_1 - 1) u_2^2 (u_2 - 1) \frac{\partial^2}{\partial u_1 \partial u_2} \widehat{\mathcal{H}}_{g-1,\ell+1} \left(u_1, u_2, t_{L \setminus \{i\}}\right) \right]_{u_1 = u_2 = t_i}$$

$$+ \frac{1}{2} \sum_{i=1}^{\ell} \sum_{\substack{g_1 + g_2 = g \\ J \sqcup K = L \setminus \{i\}}}^{\text{stable}} t_i^2 (t_i - 1) \frac{\partial}{\partial t_i} \widehat{\mathcal{H}}_{g_1, |J|+1}(t_i, t_J) \cdot t_i^2 (t_i - 1) \frac{\partial}{\partial t_i} \widehat{\mathcal{H}}_{g_2, |K|+1}(t_i, t_K).$$

In the last sum each term is restricted to satisfy the stability conditions $2g_1 - 1 + |J| > 0$ and $2g_2 - 1 + |K| > 0$.

Remark 5.2. Eqn.(5.4) is equivalent to the cut-and-join equation (3.4) and (3.14). Many other equivalent formulations have been established, including the differential equation of [15].

6. The Witten-Kontsevich theorem and the λ_g formula

It has been noticed that the asymptotic behavior of Hurwitz numbers for a large partition recovers the intersection numbers of ψ-classes [29]. Actual recovery of the Witten-Kontsevich theorem [21, 32] from the ELSV formula using this asymptotic argument is rather involved ([29], see also [20]). Since the Laplace transform contains all the information of the asymptotics, we can trivially deduce the Virasoro constraint equation, or the Dijkgraaf-Verlinde-Verlinde formula [5], for the ψ-class intersection from our main equation (1.2). In this section we observe that the *top* degree terms of the recursion is the DVV formula. We also examine that the *lowerst* degree terms imply the descendant relation of the λ_g formula [10, 11]. Our argument is along the same line with [4, 15, 18]. However, due to the polynomial formulation of (1.2), the derivation becomes simpler.

First we compute the polynomial $\hat{\xi}_n(t)$ using (4.11). It has the general form

$$\hat{\xi}_n(t) = (2n-1)!!t^{2n+1} - \frac{(2n+1)!!}{3}\, t^{2n} + \cdots + a_n t^{n+2} + (-1)^n n!\, t^{n+1}, \tag{6.1}$$

where a_n is defined by

$$a_n = -\big[(n+1)a_{n-1} + (-1)^n n!\big]$$

and is identified as the sequence A001705 or A081047 of the *On-Line Encyclopedia of Integer Sequences*.

The DVV formula for the Virasoro constraint condition on the ψ-class intersections is

$$\begin{aligned}\langle \tau_{n_L}\rangle_{g,\ell} &= \sum_{j\ge 2} \frac{(2n_1+2n_j-1)!!}{(2n_1+1)!!(2n_j-1)!!}\langle \tau_{n_1+n_j-1}\tau_{n_{L\setminus\{1,j\}}}\rangle_{g,\ell-1} \\ &+ \frac{1}{2}\sum_{a+b=n_1-2}\left(\langle \tau_a\tau_b\tau_{n_{L\setminus\{1\}}}\rangle_{g-1,\ell+1} + \sum_{\substack{g_1+g_2=g\\ J\sqcup K = L\setminus\{1\}}}^{\text{stable}} \langle \tau_a\tau_{n_J}\rangle_{g_1,|J|+1}\cdot\langle \tau_b\tau_{n_K}\rangle_{g_2,|K|+1}\right) \\ &\qquad\times \frac{(2a+1)!!(2b+1)!!}{(2n_1+1)!!}.\end{aligned} \tag{6.2}$$

Here $L = \{1,\dots,\ell\}$ is the index set as before, and for a subset $I\subset L$ we write

$$n_I = (n_i)_{i\in I} \qquad\text{and}\qquad \tau_{n_I} = \prod_{i\in I}\tau_{n_i}.$$

Proposition 6.1. *The DVV formula* (6.2) *is exactly the relation among the top degree coefficients of the recursion* (1.2).

Proof. Choose n_L so that $|n_L| = n_1+n_2+\cdots+n_\ell = 3g-3+\ell$. The degree of the LHS of (1.2) is $3(2g-2+\ell)+1$. So we compare the coefficients of $t_1^{2n_1+2}\prod_{j\ge 2} t_j^{2n_j+1}$ in the recursion formula. The contribution from the LHS of (1.2) is

$$\langle \tau_{n_L}\rangle_{g,\ell}(2n_1+1)!!\prod_{j\ge 2}(2n_j-1)!!.$$

The contribution from the first line of the RHS comes from

$$\begin{aligned}&\sum_{j\ge 2}\langle \tau_m\tau_{n_{L\setminus\{1,j\}}}\rangle_{g,\ell-1}(2m+1)!!\frac{t_1^2 t_j t_1^{2m+3} - t_j^2 t_1 t_j^{2m+3}}{t_1-t_j} \\ &\qquad = \sum_{j\ge 2}\langle \tau_m\tau_{n_{L\setminus\{1,j\}}}\rangle_{g,\ell-1}(2m+1)!!t_1t_j\frac{t_1^{2m+4}-t_j^{2m+4}}{t_1-t_j} \\ &\qquad\qquad = \sum_{j\ge 2}\langle \tau_m\tau_{n_{L\setminus\{1,j\}}}\rangle_{g,\ell-1}(2m+1)!!\sum_{a+b=2m+3} t_1^{a+1}t_j^{b+1},\end{aligned}$$

where $m = n_1+n_j-1$. The matching term in this formula is $a = 2n_1+1$ and $b = 2n_j$. Thus we extract as the coefficient of $t_1^{2n_1+2}\prod_{j\ge 2} t_j^{2n_j+1}$

$$\sum_{j\ge 2}\langle \tau_{n_1+n_j-1}\tau_{n_{L\setminus\{1,j\}}}\rangle_{g,\ell-1}(2n_1+2n_j-1)!!\prod_{k\ne 1,j}(2n_k-1)!!.$$

The contributions of the second and the third lines of the RHS of (1.2) are

$$\frac{1}{2}\sum_{a+b=n_1-2}\left(\langle\tau_a\tau_b\tau_{L\setminus\{1\}}\rangle_{g-1,\ell+1}+\frac{1}{2}\sum_{\substack{g_1+g_2=g\\ J\sqcup K=L\setminus\{1\}}}^{\text{stable}}\langle\tau_a\tau_{n_J}\rangle_{g_1,|J|+1}\cdot\langle\tau_b\tau_{n_K}\rangle_{g_2,|K|+1}\right)$$
$$\times(2a+1)!!(2b+1)!!\prod_{j\geq 2}(2n_j-1)!!.$$

We have thus recovered the Witten-Kontsevich theorem [5, 21, 32]. □

The λ_g formula [10, 11, 23, 24] is

$$\langle\tau_{n_L}\lambda_g\rangle_{g,\ell}=\binom{2g-3+\ell}{n_L}b_g, \tag{6.3}$$

where

$$\binom{2g-3+\ell}{n_L}=\binom{2g-3+\ell}{n_1,\ldots,n_\ell}$$

is the multinomial coefficient, and

$$b_g=\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}$$

is a coefficient of the series

$$\sum_{j=0}^{\infty}b_js^{2j}=\frac{s/2}{\sin(s/2)}.$$

Proposition 6.2. *The lowest degree terms of the topological recursion* (1.2) *proves the combinatorial factor of the* λ_g *formula*

$$\langle\tau_{n_L}\lambda_g\rangle_{g,\ell}=\binom{2g-3+\ell}{n_L}\langle\tau_{2g-1}\lambda_g\rangle_{g,1}. \tag{6.4}$$

Proof. Choose n_L subject to $|n_L|=2g-3+\ell$. We compare the coefficient of the terms of $\prod_{i\geq 1}t_i^{n_i+1}$ in (1.2), which has degree $|n_L|+\ell=2g-3+2\ell$. The LHS contributes

$$(-1)^{2g-3+\ell}(-1)^g\langle\tau_{n_L}\lambda_g\rangle_{g,\ell}\prod_{i\geq 1}n_i!\left(2g-2+\ell-\sum_{i=1}^{\ell}(n_i+1)\right)$$
$$=(-1)^{\ell}(-1)^g\langle\tau_{n_L}\lambda_g\rangle_{g,\ell}(\ell-1)\prod_{i\geq 1}n_i!.$$

The lowest degree terms of the first line of the RHS are

$$(-1)^g\sum_{i<j}\sum_m\langle\tau_m\tau_{L\setminus\{i,j\}}\lambda_g\rangle_{g,\ell-1}(-1)^m(m+1)!\frac{t_i^{m+4}-t_j^{m+4}}{t_i-t_j}(-1)^{2g-3+\ell-n_i-n_j}\prod_{k\neq i,j}n_k!t_k^{n_k+1}.$$

Since $m=n_i+n_j-1$, the coefficient of $\prod_{i\geq 1}t_i^{n_i+1}$ is

$$-(-1)^g(-1)^{2g-3+\ell}\sum_{i<j}\langle\tau_{n_i+n_j-1}\tau_{L\setminus\{i,j\}}\lambda_g\rangle_{g,\ell-1}\binom{n_i+n_j}{n_i}\prod_{i\geq 1}n_i!.$$

Note that the lowest degree coming from the second and the third lines of the RHS of (1.2) is $|n_L| + \ell + 2$, which is higher than the lowest degree of the LHS. Therefore, we have obtained a recursion equation with respect to ℓ

$$(\ell - 1)\langle \tau_{n_L} \lambda_g \rangle_{g,\ell} = \sum_{i<j} \langle \tau_{n_i+n_j-1} \tau_{L\setminus\{i,j\}} \lambda_g \rangle_{g,\ell-1} \binom{n_i + n_j}{n_i}. \tag{6.5}$$

The solution of the recursion equation (6.5) is the multinomial coefficient. □

Remark 6.3. Although the topological recursion (1.2) determines all linear Hodge integrals, the closed formula

$$b_g = \langle \tau_{2g-2} \lambda_g \rangle_{g,1} \qquad g \geq 1$$

does not seem to follow directly from it.

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