

THE SPECTRAL CURVE OF THE EYNARD-ORANTIN RECURSION VIA THE LAPLACE TRANSFORM

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ABSTRACT. The Eynard-Orantin recursion formula provides an effective tool for certain enumeration problems in geometry. The formula requires a *spectral curve* and the recursion kernel. We present a uniform construction of the spectral curve and the recursion kernel from the unstable geometries of the original counting problem. We examine this construction using four concrete examples: Grothendieck’s dessins d’enfants (or higher-genus analogue of the Catalan numbers), the intersection numbers of tautological cotangent classes on the moduli stack of stable pointed curves, single Hurwitz numbers, and the stationary Gromov-Witten invariants of the complex projective line.

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1. INTRODUCTION

What is the mirror dual object of the Catalan numbers? We wish to make sense of this question in the present paper. Since the homological mirror symmetry is a categorical equivalence, it does not require the existence of underlying spaces to which the categories are associated. By identifying the Catalan numbers with a counting problem similar to Gromov-Witten theory, we come up with an *equation*

$$(1.1) \quad x = z + \frac{1}{z}$$

as their mirror dual. It is not a coincidence that (1.1) is the Landau-Ginzburg model in one variable [2, 40]. Once the mirror dual object is identified, we can calculate the *higher-genus analogue* of the Catalan numbers using the *Eynard-Orantin topological recursion formula*. This recursion therefore provides a mechanism of calculating the higher-order quantum corrections term by term.

The purpose of this paper is to present a systematic construction of genus 0 spectral curves of the Eynard-Orantin recursion formula [25, 27]. Suppose we have a symplectic space X on the A-model side. If the Gromov-Witten theory of X is controlled by an integrable

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system, then the homological mirror dual of X is expected to be a family of *spectral curves* Σ . Let us consider the descendant Gromov-Witten invariants of X as a function in integer variables. The *Laplace transform* of these functions are symmetric meromorphic functions defined on the products of Σ . We expect that they satisfy the Eynard-Orantin topological recursion on the B-model side defined on the curve Σ .

More specifically, we construct the spectral curve using the Laplace transform of the descendant Gromov-Witten type invariants for the *unstable* geometries $(g, n) = (0, 1)$ and $(0, 2)$. We give four concrete examples in this paper:

- The number of dessins d'enfants of Grothendieck, which can be thought of as higher-genus analogue of the Catalan numbers.
- The ψ -class intersection numbers $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$ on the moduli space $\overline{\mathcal{M}}_{g,n}$ of pointed stable curves [10, 15, 25, 46, 78].
- Single Hurwitz numbers [7, 24, 56].
- The stationary Gromov-Witten invariants of \mathbb{P}^1 [62, 66].

The spectral curves we construct are listed in Table 1. The Eynard-Orantin recursion formula for the single Hurwitz numbers [4, 7, 24, 57] and the ψ -class intersection numbers [25] are known. Norbury and Scott conjecture that the stationary Gromov-Witten invariants of \mathbb{P}^1 also satisfy the Eynard-Orantin recursion [62]. A similar statement for the number of dessins d'enfants does not seem to be known. We give a full proof of this fact in this paper.

Grothendieck's Dessins	$\begin{cases} x = z + \frac{1}{z} \\ y = -z \end{cases}$
$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$	$\begin{cases} x = z^2 \\ y = -z \end{cases}$
Single Hurwitz Numbers	$\begin{cases} x = ze^{1-z} \\ y = e^{z-1} \end{cases}$
Stationary GW Invariants of \mathbb{P}^1	$\begin{cases} x = z + \frac{1}{z} \\ y = -\log(1 + z^2) \end{cases}$

TABLE 1. Examples of spectral curves.

Let $D_{g,n}(\mu_1, \dots, \mu_n)$ denote the weighted count of clean Belyi morphisms of smooth connected algebraic curves of genus g with n poles of order (μ_1, \dots, μ_n) . We first prove

Theorem 1.1. *For $2g - 2 + n \geq 0$ and $n \geq 1$, the number of clean Belyi morphisms satisfies the following equation:*

$$(1.2) \quad \mu_1 D_{g,n}(\mu_1, \dots, \mu_n) = \sum_{j=2}^n (\mu_1 + \mu_j - 2) D_{g,n-1}(\mu_1 + \mu_j - 2, \mu_{[n] \setminus \{1,j\}}) \\ + \sum_{\alpha+\beta=\mu_1-2} \alpha\beta \left[D_{g-1,n+1}(\alpha, \beta, \mu_{[n] \setminus \{1\}}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2, \dots, n\}}} D_{g_1, |I|+1}(\alpha, \mu_I) D_{g_2, |J|+1}(\beta, \mu_J) \right],$$

where $\mu_I = (\mu_i)_{i \in I}$ for a subset $I \subset [n] = \{1, 2, \dots, n\}$.

The simplest case

$$D_{0,1}(2m) = \frac{1}{2m} C_m$$

is given by the Catalan number $C_m = \frac{1}{m+1} \binom{2m}{m}$. The next case $D_{0,2}(\mu_1, \mu_2)$ is calculated in [44, 45]. Note that the (g, n) -terms appears also on the right-hand side of (1.2). Therefore, this is merely an equation, not an effective recursion formula.

Define the Eynard-Orantin differential form by

$$W_{g,n}^D(t_1, \dots, t_n) = d_1 \cdots d_n \sum_{\mu_1, \dots, \mu_n > 0} D_{g,n}(\mu_1, \dots, \mu_n) e^{-(\mu_1 w_1 + \cdots + \mu_n w_n)},$$

where the w_j -coordinates and t_j -coordinates are related by

$$e^{w_j} = \frac{t_j + 1}{t_j - 1} + \frac{t_j - 1}{t_j + 1}.$$

Then

Theorem 1.2. *The Eynard-Orantin differential forms for $2g - 2 + n > 0$ satisfy the following topological recursion formula*

$$(1.3) \quad W_{g,n}^D(t_1, \dots, t_n) \\ = -\frac{1}{64} \frac{1}{2\pi i} \int_{\gamma} \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \cdot \frac{1}{dt} \cdot dt_1 \left[W_{g-1, n+1}^D(t, -t, t_2, \dots, t_n) \right. \\ \left. + \sum_{j=2}^n \left(W_{0,2}^D(t, t_j) W_{g, n-1}^D(-t, t_2, \dots, \hat{t}_j, \dots, t_n) + W_{0,2}^D(-t, t_j) W_{g, n-1}^D(t, t_2, \dots, \hat{t}_j, \dots, t_n) \right) \right. \\ \left. + \sum_{\substack{I \sqcup J = \{2, 3, \dots, n\} \\ g_1 + g_2 = g}}^{stable} W_{g_1, |I|+1}^D(t, t_I) W_{g_2, |J|+1}^D(-t, t_J) \right].$$

This is now a recursion formula, since the topological type (g', n') of the Belyi morphisms appearing on the right-hand side satisfies

$$2g' - 2 + n' = (2g - 2 + n) - 1,$$

counting the contributions from the disjoint union of the domain curves additively. A corollary to the recursion formula is a combinatorial identity between the number of clean Belyi morphisms and the number of lattice points on the moduli space $\mathcal{M}_{g,n}$ that has been studied in [10, 55, 59, 60, 61].

Corollary 1.3.

$$(1.4) \quad D_{g,n}(\mu_1, \dots, \mu_n) = \sum_{\ell_1 > \frac{\mu_1}{2}} \cdots \sum_{\ell_n > \frac{\mu_n}{2}} \prod_{i=1}^n \frac{2\ell_i - \mu_i}{\mu_i} \binom{\mu_i}{\ell_i} N_{g,n}(2\ell_1 - \mu_1, \dots, 2\ell_n - \mu_n),$$

where $N_{g,n}(\mu_1, \dots, \mu_n)$ is defined by (4.18).

The recursion formula (1.3) is a typical example of the *Eynard-Orantin recursion* we discuss in this paper. We establish this theorem by taking the Laplace transform of (1.2). This is indeed a general theme. For every known case of the Eynard-Orantin recursion, we establish its proof by taking the Laplace transform of the counting formula like (1.2). For example, for the cases of single Hurwitz numbers [24, 57] and open Gromov-Witten

invariants of \mathbb{C}^3 [81, 82], the counting formulas similar to (1.2) are called the *cut-and-join* equations [29, 77, 49, 79, 80].

The Laplace transform plays a mysterious role in Gromov-Witten theory. We notice its appearance in Kontsevich's work [46] that relates the Euclidean volume of $\mathcal{M}_{g,n}$ and the intersection numbers on $\overline{\mathcal{M}}_{g,n}$, and also in the work of Okounkov-Pandharipande [65] that relates the single Hurwitz numbers and the enumeration of topological graphs. It has been proved that in these two cases the Laplace transform of the quantities in question satisfies the Eynard-Orantin recursion [10, 24, 26, 55, 57] for a particular choice of the spectral curve.

Then what is the role of the Laplace transform here? The answer we propose in this paper is that *the Laplace transform defines the spectral curve*. Since the spectral curve is a B-model object, **the Laplace transform plays the role of mirror symmetry**.

The Eynard-Orantin recursion formula is an effective tool in certain geometric enumeration. The formula is originated in random matrix theory as a machinery to compute the expectation value of a product of the resolvent of random matrices ([1], [21]). In [25, 27] Eynard and Orantin propose a novel point of view, considering the recursion as a mechanism of *defining* meromorphic symmetric differential forms $W_{g,n}$ on the product Σ^n of a Riemann surface Σ for every $g \geq 0$ and $n > 0$. They derive in [25, 27] many beautiful properties that these quantities satisfy, including modularity and relations to integrable systems.

The effectiveness of the topological recursion in string theory is immediately noticed [14, 23, 51, 70]. A remarkable discovery, connecting the recursion formula and geometry, is made by Mariño [51] and Bouchard, Klemm, Mariño and Pasquetti [6]. It is formulated as the *Remodeling Conjecture*. This conjecture covers many aspects of both closed and open Gromov-Witten invariants of arbitrary toric Calabi-Yau threefolds. One of their statements says the following. Let X be an arbitrary toric Calabi-Yau threefold, and Σ its mirror curve. Apply the Eynard-Orantin recursion formula to Σ . Then $W_{g,n}$ calculates the open Gromov-Witten invariants of X . The validity of the topological recursion of [25, 27] is not limited to Gromov-Witten invariants. It has been applied to the HOMFLY polynomials of torus knots [9], and understanding the role of quantum Riemann surfaces and certain Seiberg-Witten invariants [35]. A speculation also suggests its relation to colored Jones polynomials and the hyperbolic volume conjecture of knot complements [13].

From the very beginning, effectiveness of the Eynard-Orantin recursion in enumerative geometry was suggested by physicists. Bouchard and Mariño conjecture in [7] that particular generating functions of single Hurwitz numbers satisfy the Eynard-Orantin topological recursion. They have come up to this conjecture as the limiting case of the remodeling conjecture for \mathbb{C}^3 when the framing parameter tends to ∞ . The spectral curve for this scenario is the *Lambert curve* $x = ye^{-y}$. The Bouchard-Mariño conjecture is solved in [4, 24, 57]. The work [24] also influenced the solutions to the remodeling conjecture for \mathbb{C}^3 itself. The statement on the open Gromov-Witten invariants was proved in [11, 81, 82], and the closed case was proved in [5, 83].

The Eynard-Orantin topological recursion *starts* with a spectral curve Σ . Thus it is reasonable to propose the recursion formalism whenever there is a natural curve in the problem we study. Such curves may include the mirror curve of a toric Calabi-Yau threefold [6, 51], the zero locus of an A-polynomial [13, 35], the Seiberg-Witten curves [35], the torus on which a knot is drawn [9], and the character variety of the fundamental group of a knot complement relative to $SL(2, \mathbb{C})$ [13]. Now we ask the opposite question.

Question 1.4. *If an enumerative geometry problem is given, then how do we find the spectral curve, with which the Eynard-Orantin formalism may provide a solution?*

In every work of [5, 10, 11, 24, 25, 26, 27, 55, 57, 60, 62, 81, 82], the spectral curve is considered to be given. *How do we know that the particular choice of the spectral curve is correct?* Our proposal provides an answer to this question: *the Laplace transform of the unstable geometries $(g, n) = (0, 1)$ and $(0, 2)$ determines the spectral curve, and the topological recursion formula itself.* The key ingredients of the topological recursion are the spectral curve and the recursion kernel that is determined by the differential forms $W_{0,1}$ and $W_{0,2}$. In the literature starting from [25], the word “Bergman kernel” is used for the differential form $W_{0,2}$. But it has indeed nothing to do with the classical *Bergman kernel* in complex analysis. It is also treated as the universally given 2-form depending only on the geometry of the spectral curve. We would rather emphasize in this paper that this “kernel” is the Laplace transform of the *annulus* amplitude, which should be determined by the counting problem we start with.

Although it is still vague, our proposal is the following

Conjecture 1.5 (The Laplace transform conjecture). *If the unstable geometries $(g, n) = (0, 1)$ and $(0, 2)$ make sense in a counting problem on the A -model side, then the Laplace transform of the solution to these cases determines the spectral curve and the recursion kernel of the Eynard-Orantin formalism, which is a B -model theory. Thus the Laplace transform plays a role of mirror symmetry. The recursion then determines the solution to the original counting problem for all (g, n) .*

The Eynard-Orantin recursion is a process of *quantization* [9, 35]. Thus the implication of the conjecture is that *quantum invariants are uniquely determined by the disk and annulus amplitudes.* For example, single Hurwitz numbers $h_{g,\mu}$ are all determined by the first two cases $h_{0,(\mu_1)}$ and $h_{0,(\mu_1,\mu_2)}$. The present paper and our previous work [24, 57] establish this fact. The Lambert curve is the mirror dual of the number of trees.

The organization of this paper is the following. In Section 2 we present the Eynard-Orantin recursion formalism for the case of genus 0 spectral curve. Higher genus situations will be discussed elsewhere. Sections 3 and 4 deal with the counting problem of Grothendieck’s dessins d’enfants. We present our new results on this problem, which are Theorem 1.1 and Theorem 1.2. We are inspired by Kodama’s beautiful talk [44] (that is based on [45]) to come up with the generating function of the Catalan numbers as the spectral curve for this problem. We are grateful to G. Gliner for drawing our attention to [44]. The counting problem of the lattice points on $\mathcal{M}_{g,n}$ of [10, 55, 59, 60] is closely related to the counting of dessins, which is also treated in Section 4. The Eynard-Orantin recursion becomes identical to the Virasoro constraint condition for the ψ -class intersection numbers on $\overline{\mathcal{M}}_{g,n}$. We discuss this relation in Section 5, using Kontsevich’s idea that the intersection numbers on $\overline{\mathcal{M}}_{g,n}$ are essentially the same as Euclidean volume of $\mathcal{M}_{g,n}$. Section 6 is devoted to single Hurwitz numbers. In our earlier work [24, 57] we used the Lambert curve as given. Here we reexamine the Hurwitz counting problem and derive the Lambert curve from the unstable geometries. We then consider the Norbury-Scott conjecture [62] in Section 7, which states that the generating functions of stationary Gromov-Witten invariants of \mathbb{P}^1 satisfy the Eynard-Orantin recursion. We are unable to prove this conjecture. What we establish in this section is *why* the spectral curve of [62] is the right choice for this problem.

The subject of this paper is closely related to random matrix theory. Since the matrix model side of the story has been extensively discussed by the original authors [27], we do not deal with that aspect in the current paper.

2. THE EYNARD-ORANTIN DIFFERENTIAL FORMS AND THE TOPOLOGICAL RECURSION

We use the following mathematical definition for the topological recursion of Eynard-Orantin for a genus 0 spectral curve. The differences between our definition and the original formulation found in [25, 27] are of the philosophical nature. Indeed, the original formula and ours produce the exact same answer in all examples we examine in this paper.

Definition 2.1. We start with \mathbb{P}^1 with a *preferred coordinate* t . Let $S \subset \mathbb{P}^1$ be a finite collection of points and compact real curves such that the complement $\Sigma = \mathbb{P}^1 \setminus S$ is connected. The *spectral curve* of genus 0 is the data (Σ, π) consisting of a Riemann surface Σ and a simply ramified holomorphic map

$$(2.1) \quad \pi : \Sigma \ni t \longmapsto \pi(t) = x \in \mathbb{P}^1$$

so that its differential dx has only simple zeros. Let us denote by $R = \{p_1, \dots, p_r\} \subset \Sigma$ the ramification points, and by

$$U = \sqcup_{j=1}^r U_j$$

the disjoint union of small neighborhood U_j around each p_j such that $\pi : U_j \rightarrow \pi(U_j) \subset \mathbb{P}^1$ is a double-sheeted covering ramified only at p_j . We denote by $\bar{t} = s(t)$ the local Galois conjugate of $t \in U_j$. The canonical sheaf of Σ is denoted by \mathcal{K} . Because of our choice of the preferred coordinate t , we have a preferred basis dt for \mathcal{K} and $\partial/\partial t$ for \mathcal{K}^{-1} . The meromorphic differential forms $W_{g,n}(t_1, \dots, t_n)$, $g = 0, 1, 2, \dots, n = 1, 2, 3, \dots$, are said to satisfy the **Eynard-Orantin topological recursion** if the following conditions are satisfied:

- (1) $W_{0,1}(t) \in H^0(\Sigma, \mathcal{K})$.
- (2) $W_{0,2}(t_1, t_2) = \frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2} - \pi^* \frac{dx_1 \cdot dx_2}{(x_1 - x_2)^2} \in H^0(\Sigma \times \Sigma, \mathcal{K}^{\otimes 2}(2\Delta))$, where Δ is the diagonal of $\Sigma \times \Sigma$.
- (3) The recursion kernel $K_j(t, t_1) \in H^0(U_j \times C, (\mathcal{K}_{U_j}^{-1} \otimes \mathcal{K})(\Delta))$ for $t \in U_j$ and $t_1 \in C$ is defined by

$$(2.2) \quad K_j(t, t_1) = \frac{1}{2} \frac{\int_t^{\bar{t}} W_{0,2}(\cdot, t_1)}{W_{0,1}(\bar{t}) - W_{0,1}(t)}.$$

The kernel is an algebraic operator that multiplies dt_1 while contracts dt .

- (4) The general differential forms $W_{g,n}(t_1, \dots, t_n) \in H^0(\Sigma^n, \mathcal{K}(*R)^{\otimes n})$ are meromorphic symmetric differential forms with poles only at the ramification points R for $2g - 2 + n > 0$, and are given by the recursion formula

$$(2.3) \quad W_{g,n}(t_1, t_2, \dots, t_n) = \frac{1}{2\pi i} \sum_{j=1}^r \oint_{U_j} K_j(t, t_1) \left[W_{g-1, n+1}(t, \bar{t}, t_2, \dots, t_n) \right. \\ \left. + \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = \{2, 3, \dots, n\}}} \text{No } (0, 1) \text{ terms} \right. \\ \left. W_{g_1, |I|+1}(t, t_I) W_{g_2, |J|+1}(\bar{t}, t_J) \right].$$

Here the integration is taken with respect to $t \in U_j$ along a positively oriented simple closed loop around p_j , and $t_I = (t_i)_{i \in I}$ for a subset $I \subset \{1, 2, \dots, n\}$.

- (5) The differential form $W_{1,1}(t_1)$ requires a separate treatment because $W_{0,2}(t_1, t_2)$ is regular at the ramification points but has poles elsewhere.

$$(2.4) \quad W_{1,1}(t_1) = \frac{1}{2\pi i} \sum_{j=1}^r \oint_{U_j} K_j(t, t_1) \left[W_{0,2}(u, v) + \pi^* \frac{dx(u) \cdot dx(v)}{(x(u) - x(v))^2} \right] \Bigg|_{\substack{u=t \\ v=\bar{t}}}.$$

Let $y : \Sigma \rightarrow \mathbb{C}$ be a holomorphic function defined by the equation

$$(2.5) \quad W_{0,1}(t) = y(t)dx(t).$$

Equivalently, we can define the function by contraction $y = i_{\mathcal{X}}W_{0,1}$, where \mathcal{X} is the vector field on Σ dual to $dx(t)$ with respect to the coordinate t . Then we have an embedding

$$\Sigma \ni t \mapsto (x(t), y(t)) \in \mathbb{C}^2.$$

Remark 2.2. The recursion (2.3) also applies to $(g, n) = (0, 3)$, which gives $W_{0,3}$ in terms of $W_{0,2}$. In [25, Theorem 4.1] an equivalent but often more useful formula for $W_{0,3}$ is given:

$$(2.6) \quad W_{0,3}(t_1, t_2, t_3) = \frac{1}{2\pi i} \sum_{j=1}^r \oint_{U_j} \frac{W_{0,2}(t, t_1)W_{0,2}(t, t_2)W_{0,2}(t, t_3)}{dx(t) \cdot dy(t)}.$$

3. COUNTING GROTHENDIECK'S DESSINS D'ENFANTS

The A-model side of the problem we consider in this section is the counting problem of Grothendieck's *dessins d'enfants* (see for example, [71, 72]) for a fixed topological type of Belyi morphisms [3]. Gromov-Witten theory of an algebraic variety X is an intersection theory of naturally defined divisors on the moduli stack $\overline{\mathcal{M}}_{g,n}(X)$ of stable morphisms from n -pointed algebraic curves of genus g to the target variety X . Since we are considering tautological divisors, their 0-dimensional intersection points are also *natural*. These points determine a finite set on $\overline{\mathcal{M}}_{g,n}$ via the stabilization morphism. If we expect that the Gromov-Witten theory of X satisfies the Eynard-Orantin recursion, then we should also expect that the counting problem of naturally defined finite sets of points on $\overline{\mathcal{M}}_{g,n}$ may satisfy the Eynard-Orantin recursion.

Pointed curves defined over $\overline{\mathbb{Q}}$ form a dense subset of $\overline{\mathcal{M}}_{g,n}$. To specify n , we need to use Belyi morphisms. When we identify a curve over $\overline{\mathbb{Q}}$ with a Belyi morphism, a natural counting problem arises by considering the profile of the Belyi morphism at the branched points. In this way we arrive at canonically defined finite sets of points on $\overline{\mathcal{M}}_{g,n}$.

More specifically, consider a Belyi morphism

$$(3.1) \quad b : C \rightarrow \mathbb{P}^1$$

of a smooth algebraic curve C of genus g . This means b is branched only over $0, 1, \infty \in \mathbb{P}^1$. By Belyi's Theorem [3], C is defined over $\overline{\mathbb{Q}}$. Let q_1, \dots, q_n be poles of b of orders $(\mu_1, \dots, \mu_n) \in \mathbb{Z}_+^n$. This vector of positive integers is the *profile* of b at ∞ . In our enumeration we *label* all poles of b . Therefore, an automorphism of a Belyi morphism preserves the set of poles point-wise.

A *clean* Belyi morphism is a special class of Belyi morphism of even degree that has profile $(2, 2, \dots, 2)$ over the branch point $1 \in \mathbb{P}^1$. We note that a complex algebraic curve is defined over $\overline{\mathbb{Q}}$ if and only if it admits a clean Belyi morphism. Let us denote by $D_{g,n}(\mu_1, \dots, \mu_n)$ the number of genus g clean Belyi morphisms of profile (μ_1, \dots, μ_n) at $\infty \in \mathbb{P}^1$. This is the number we study in this section.

We first derive a recursion equation among $D_{g,n}(\mu_1, \dots, \mu_n)$ for all (g, n) . This relation does not provide an effective recursion formula, because $D_{g,n}(\mu_1, \dots, \mu_n)$ appears in the equation in a complicated manner. We then compute the Laplace transform

$$F_{g,n}^D(w_1, \dots, w_n) = \sum_{\mu_1, \dots, \mu_n > 0} D_{g,n}(\mu_1, \dots, \mu_n) e^{-(\mu_1 w_1 + \dots + \mu_n w_n)},$$

and rewrite the recursion equation in terms of the Laplace transformed functions. We then show that the symmetric differential forms

$$W_{g,n}^D = d_1 \cdots d_n F_{g,n}^D$$

satisfy the Eynard-Orantin recursion formula. This time it is an effective recursion formula for the generating functions $W_{g,n}^D$ of the number $D_{g,n}(\mu_1, \dots, \mu_n)$ of clean Belyi morphisms.

Grothendieck visualized the clean Belyi morphism by considering the inverse image

$$(3.2) \quad \Gamma = b^{-1}([0, 1])$$

of the closed interval $[0, 1] \subset \mathbb{P}^1$ by b (see his “Esquisse d’un programme” reprinted in [72]). This is what we call *dessin d’enfant*. It is a topological graph drawn on the algebraic curve C being considered as a Riemann surface. We call each pre-image of $0 \in \mathbb{P}^1$ by b a *vertex* of Γ . Since b has profile $(2, \dots, 2)$ over $1 \in \mathbb{P}^1$, a pre-image of 1 is the midpoint of an *edge* of Γ . The complement $C \setminus \Gamma$ of Γ in C is the disjoint union of n disks centered at each q_i . By abuse of terminology we call each disk a *face* of Γ . Then by Euler’s formula we have

$$2 - 2g = |b^{-1}(0)| - |b^{-1}(1)| + n.$$

A dessin is a special kind of metric ribbon graph. A *ribbon graph* of topological type (g, n) is the 1-skeleton of a cell-decomposition of a closed oriented topological surface C of genus g that decomposes the surface into a disjoint union of 0-cells, 1-cells, and 2-cells. The number of 2-cells is n . Alternatively, a ribbon graph can be defined as a graph with a cyclic order assigned to the incident half-edges at each vertex. When a positive real number, the length, is assigned to each edge of a ribbon graph, we call it a *metric* ribbon graph. A dessin is thus a metric ribbon graph with the same length given to each edge. We usually consider this length to be 1, so the distance between 0 and 1 on \mathbb{P}^1 is measured as $\frac{1}{2}$.

The concrete construction of [54] gives a Belyi morphism to any given dessin. Thus the enumeration of clean Belyi morphism is equivalent to the enumeration of ribbon graphs, where we assign length 1 to every edge. The original interest of dessins lies in the fact that the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of dessins.

An alternative description of a Belyi morphism is to use the dual graph

$$(3.3) \quad \check{\Gamma} = b^{-1}([1, i\infty]),$$

where

$$[1, i\infty] = \{1 + iy \mid 0 \leq y \leq \infty\} \subset \mathbb{P}^1$$

is the vertical half-line on \mathbb{P}^1 with real part 1. This time the graph $\check{\Gamma}$ has n labeled vertices of degrees (μ_1, \dots, μ_n) . Since we consider ribbon graphs in the context of canonical cell-decomposition of the moduli space $\mathcal{M}_{g,n}$, we use the terminology *dessin* for a graph $\check{\Gamma}$ dual to a ribbon graph Γ . This distinction is important, because when we count the number of ribbon graphs, we consider the automorphism of a graph that preserves each face, while the automorphism group of the dual graph, i.e., a dessin, preserves each vertex point-wise, but can permute faces. In this dual picture, we define the number of dessins with the automorphism factor by

$$(3.4) \quad D_{g,n}(\mu_1, \dots, \mu_n) = \sum_{\check{\Gamma} \text{ dessin of type } (g,n)} \frac{1}{|\text{Aut}_D(\check{\Gamma})|},$$

where $\check{\Gamma}$ is a dessin of genus g with n labeled vertices with prescribed degrees (μ_1, \dots, μ_n) , and $\text{Aut}_D(\check{\Gamma})$ is the automorphism of $\check{\Gamma}$ preserving each vertex point-wise.

Our theme is to find the spectral curve of the theory by looking at the problem for unstable curves $(g, n) = (0, 1)$ and $(0, 2)$. The dessins counted in $D_{0,1}(\mu)$ for an integer $\mu \in \mathbb{Z}_+$ are spherical graphs that contain only one vertex of degree μ . Since any edge of this graph has to start and end with the same vertex, it is a loop, and thus μ is even. So let us put $\mu = 2m$. Each graph contributes with the weight $1/|\text{Aut}_D(\check{\Gamma})|$ in the enumeration of the number $D_{0,1}(\mu)$. This automorphism factor makes counting more difficult. Note that the automorphism group of a spherical dessin with a single vertex is a subgroup of $\mathbb{Z}/(2m)\mathbb{Z}$ that preserves the graph. If we place an outgoing arrow to one of the $2m$ half-edges incident to the unique vertex (see Figure 3.1), then we can kill the automorphism altogether. Since there are $2m$ choices of placing such an arrow, the number of arrowed graphs is $2mD_{0,1}(2m)$. This is now an integer. By a simple bijection argument with the number of arrangements of m pairs of parentheses, we see that

$$(3.5) \quad 2mD_{0,1}(2m) = C_m = \frac{1}{m+1} \binom{2m}{m},$$

where C_m is the m -th *Catalan number*. We note that the Catalan numbers appear in the same context of counting graphs in [37].

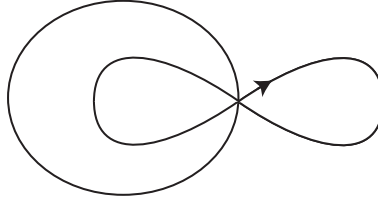


FIGURE 3.1. An arrowed dessin d'enfant of genus 0 with one vertex.

Define the Laplace transform of $D_{0,1}(\mu)$ by

$$(3.6) \quad \tilde{F}_{0,1}^D = \sum_{m=1}^{\infty} D_{0,1}(2m) e^{-2mw}.$$

Then the Eynard-Orantin differential

$$\tilde{W}_{0,1}^D = d\tilde{F}_{0,1}^D = - \sum_{m=1}^{\infty} 2mD_{0,1}(2m) e^{-2mw} dw = - \sum_{m=1}^{\infty} C_m e^{-2mw} dw$$

is a generating function of the Catalan numbers. Actually a better choice is (see [44, 45])

$$(3.7) \quad z(x) = \sum_{m=0}^{\infty} C_m \frac{1}{x^{2m+1}} = \frac{1}{x} + \frac{1}{x^3} + \frac{2}{x^5} + \frac{5}{x^7} + \frac{14}{x^9} + \frac{42}{x^{11}} + \dots$$

The radius of convergence of this infinite Laurent series is 2, hence the series converges absolutely for $|x| > 2$. The inverse function of $z = z(x)$ near $(x, z) = (\infty, 0)$ is given by

$$(3.8) \quad x = z + \frac{1}{z}.$$

This can be easily seen by solving the quadratic equation $z^2 - xz + 1 = 0$ with respect to z , which is equivalent to the quadratic recursion

$$C_{m+1} = \sum_{i+j=m} C_i \cdot C_j$$

of Catalan numbers. To take advantage of these simple formulas, let us *define*

$$(3.9) \quad x = e^w$$

and allow the $m = 0$ term in the Eynard-Orantin differential:

$$(3.10) \quad W_{0,1}^D = - \sum_{m=0}^{\infty} C_m \frac{dx}{x^{2m+1}}.$$

Accordingly the Laplace transform of $D_{0,1}(2m)$ needs to be modified:

$$(3.11) \quad F_{0,1}^D = \sum_{m=1}^{\infty} D_{0,1}(2m) e^{-2mw} - w = \sum_{m=1}^{\infty} D_{0,1}(2m) \frac{1}{x^{2m}} - \log x.$$

Although numerically $D_{0,1}(0) = 0$, its infinitesimal behavior is given by

$$\lim_{m \rightarrow 0} \frac{D_{0,1}(2m)}{x^{2m}} = -\log x,$$

which is consistent with

$$\lim_{m \rightarrow 0} 2m D_{0,1}(2m) = C_0 = 1.$$

From (3.7) and (3.12), we obtain

$$(3.12) \quad W_{0,1}^D = -z(x) dx.$$

In light of (2.5), we have identified the spectral curve for the counting problem of dessins $D_{g,n}(\mu)$. It is given by

$$(3.13) \quad \begin{cases} x = z + \frac{1}{z} \\ y = -z \end{cases}.$$

To compute the recursion kernel of (2.2), we need to identify $D_{0,2}(\mu_1, \mu_2)$ for the other unstable geometry $(g, n) = (0, 2)$. In the dual graph picture, $D_{0,2}(\mu_1, \mu_2)$ counts the number of spherical dessins $\check{\Gamma}$ with two vertices of degree μ_1 and μ_2 , counted with the weight of $1/|\text{Aut}_D(\check{\Gamma})|$. The computation was done by Kodama and Pierce in [45, Theorem 3.1]. We also refer to a beautiful lecture by Kodama [44].

Proposition 3.1 ([45]). *The number of spherical dessins $\check{\Gamma}$ with two vertices of degrees j and k , counted with the weight of $1/|\text{Aut}_D(\check{\Gamma})|$, is given by the following formula.*

$$(3.14) \quad D_{0,2}(\mu_1, \mu_2) = \begin{cases} \frac{1}{2k} \binom{2k}{k} & \mu_1 = 0, \mu_2 = 2k \neq 0 \\ \frac{1}{4} \frac{1}{j+k} \binom{2j}{j} \binom{2k}{k} & \mu_1 = 2j \neq 0, \mu_2 = 2k \neq 0 \\ \frac{1}{j+k+1} \binom{2j}{j} \binom{2k}{k} & \mu_1 = 2j + 1, \mu_2 = 2k + 1 \end{cases}.$$

All other cases $D_{0,2}(\mu_1, \mu_2) = 0$. Here the automorphism group $\text{Aut}_D(\check{\Gamma})$ is the topological graph automorphisms that fix each vertex, but may permute faces.

Remark 3.2. The first case is irregular. For $\mu_1 = 0$, the second vertex has an even degree, and hence we have $C_k/(2k)$ graphs. Note that this graph has $k + 1$ faces due to Euler's formula $2 = 1 - k + (k + 1)$. The degree 0 vertex has to be placed in one of these faces, which makes the total number of graphs

$$\frac{k+1}{2k} C_k = \frac{1}{2k} \binom{2k}{k}.$$

However, we are counting only connected graphs. Hence degree 0 vertices are not allowed in our counting.

In general the number of dessins satisfies the following:

Theorem 3.3. *For $g \geq 0$ and $n \geq 1$ subject to $2g - 2 + n \geq 0$, the number of dessins (3.4) satisfies a recursion equation*

$$(3.15) \quad \mu_1 D_{g,n}(\mu_1, \dots, \mu_n) = \sum_{j=2}^n (\mu_1 + \mu_j - 2) D_{g,n-1}(\mu_1 + \mu_j - 2, \mu_{[n] \setminus \{1,j\}}) \\ + \sum_{\alpha+\beta=\mu_1-2} \alpha\beta \left[D_{g-1,n+1}(\alpha, \beta, \mu_{[n] \setminus \{1\}}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2, \dots, n\}}} D_{g_1, |I|+1}(\alpha, \mu_I) D_{g_2, |J|+1}(\beta, \mu_J) \right],$$

where $\mu_I = (\mu_i)_{i \in I}$ for a subset $I \subset [n] = \{1, 2, \dots, n\}$. The last sum is over all partitions of the genus g and the index set $\{2, 3, \dots, n\}$ into two pieces.

Remark 3.4. Note that when $g_1 = 0$ and $I = \emptyset$, $D_{g,n}$ appears in the right-hand side of (3.15). Therefore, this is an equation of the number of dessins, not a recursion formula.

Proof. Consider the collection of genus g dessins with n vertices labeled by the index set $[n] = \{1, 2, \dots, n\}$ and of degrees (μ_1, \dots, μ_n) . The left-hand side of (3.15) is the number of dessins with an outward arrow placed on one of the incident edges at the vertex 1. The equation is based on the removal of this edge. There are two cases.

Case 1. The arrowed edge connects the vertex 1 and vertex $j > 1$. We then remove the edge and put the two vertices 1 and j together as shown in Figure 3.2. This operation is better described as shrinking the arrowed edge to a point. The resulting dessin has one less vertices, but the genus is the same as before. The degree of the newly created vertex is $\mu_1 + \mu_j - 2$, while the degrees of all other vertices are unaffected.

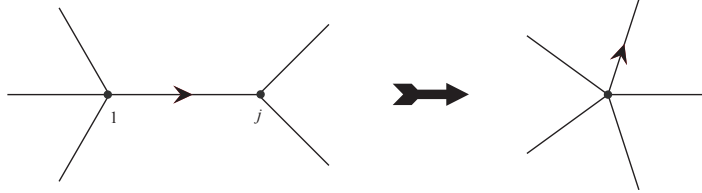


FIGURE 3.2. The operation that shrinks the arrowed edge to a point and joins two vertices labeled by 1 and j together.

To make the bijection argument, we need to be able to reconstruct the original dessin from the new one. Since both μ_1 and μ_j are given as the input value, we have to specify which edges go to vertex 1 and which go to j when we separate the vertex of degree $\mu_1 + \mu_j - 2$. For this purpose, what we need is a marker on one of the incident edges. We group the marked edge and $\mu_i - 2$ edges following it according to the cyclic order. The rest of the $\mu_j - 1$ incident edges are also grouped. Then we insert an edge and separate the vertex into two vertices, 1 and j , so that the first group of edges are incident to vertex 1 and the second group is incident to j , honoring their cyclic orders (see Figure 3.2). The contribution from this case is therefore

$$\sum_{j=2}^n (\mu_1 + \mu_j - 2) D_{g,n-1}(\mu_1 + \mu_j - 2, \mu_{[n] \setminus \{1,j\}}).$$

Case 2. The arrowed edge forms a loop that is attached to vertex 1. We remove this loop from the dessin, and separate the vertex into two vertices. The loop classifies all incident half-edges, except for the loop itself, into two groups: the ones that follow the arrowed half-edge in the cyclic order but before the incoming end of the loop, and all others (see Figure 3.3). Let α be the number of half-edges in the first group, and β the rest. Then $\alpha + \beta = \mu_1 - 2$, and we have created two vertices of degrees α and β .

To recover the original dessin from the new one, we need to mark a half-edge from each vertex so that we can put the loop back to the original place. The number of choices of these markings is $\alpha\beta$.

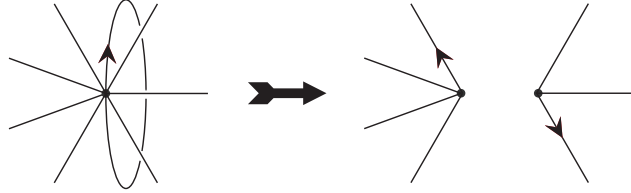


FIGURE 3.3. The operation that removes a loop, and separates the incident vertex into two vertices.

The operation of the removal of the loop and the separation of the vertex into two vertices certainly increases the number of vertices from n to $n + 1$. This operation also affects the genus of the dessin. If the resulting dessin is connected, then g goes down to $g - 1$. If the result is the disjoint union of two dessins of genera g_1 and g_2 , then we have $g = g_1 + g_2$. Altogether the contribution from this case is

$$\sum_{\alpha+\beta=\mu_1-2} \alpha\beta \left[D_{g-1, n+1}(\alpha, \beta, \mu_{[n]\setminus\{1\}}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2, \dots, n\}}} D_{g_1, |I|+1}(\alpha, \mu_I) D_{g_2, |J|+1}(\beta, \mu_J) \right].$$

Note that the outward arrow we place defines the two groups of incident half-edges uniquely, since one is after and the other before the arrowed half-edge according to the cyclic order. Thus we do not need to symmetrize α and β . Indeed, if the arrow is placed in the other end of the loop, then α and β are interchanged.

The right-hand side of the equation (3.15) is the sum of the above two contributions. \square

Remark 3.5. The equation (3.15) is considerably simpler, compared to the recursion formula for the number of ribbon graphs with integral edge lengths that is proved in [10, Theorem 3.3]. The edge removal operation of [10] is the dual operation of the edge shrinking operations of Case 1 and Case 2 above, and the placement of an arrow corresponds to the *ciliation* of [10]. In the dual picture, the graphs enumerated in [10] are more restrictive than arbitrary clean dessins, which makes the equation more complicated. We also note that [10, Theorem 3.3] is a recursion formula, not just a mere relation like what we have in (3.15). In this regard, (3.15) is indeed similar to the cut-and-join equation (6.28) of [29, 77]. We will come back to this point in Section 6.

The relation (3.15) becomes an effective recursion formula after taking the Laplace transform.

4. THE LAPLACE TRANSFORM OF THE NUMBER OF DESSINS AND RIBBON GRAPHS

In this section we derive the Eynard-Orantin recursion formula for the generating functions of the number of dessins. The key technique is the Laplace transform.

Since the projection $x = z + 1/z$ of the spectral curve to the x -coordinate plane has two ramification points $z = \pm 1$, it is natural to introduce a coordinate that has these ramification points at 0 and ∞ . So we define

$$(4.1) \quad z = \frac{t+1}{t-1}.$$

Proposition 4.1. *The Laplace transform of $D_{0,2}(\mu_1, \mu_2)$ is given by*

$$(4.2) \quad F_{0,2}^D(t_1, t_2) \stackrel{\text{def}}{=} \sum_{\mu_1, \mu_2 > 0} D_{0,2}(\mu_1, \mu_2) e^{-(\mu_1 w_1 + \mu_2 w_2)} = -\log(1 - z(x_1)z(x_2)) \\ = \log(t_1 - 1) + \log(t_2 - 1) - \log(-2(t_1 + t_2)),$$

where $z(x)$ is the generating function of the Catalan numbers (3.7), and the variables t, w, x, z are related by (3.9), (3.13), and (4.1). We then have

$$(4.3) \quad W_{0,2}^D(t_1, t_2) = d_1 d_2 F_{0,2}^D(t_1, t_2) = \frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2} - \frac{dx_1 \cdot dx_2}{(x_1 - x_2)^2} = \frac{dt_1 \cdot dt_2}{(t_1 + t_2)^2}.$$

Proof. In terms of $x = e^w$, the Laplace transform (4.2) is given by

$$(4.4) \quad \sum_{\mu_1, \mu_2 > 0} D_{0,2}(\mu_1, \mu_2) e^{-(\mu_1 w_1 + \mu_2 w_2)} \\ = \frac{1}{4} \sum_{j,k=1}^{\infty} \frac{1}{j+k} \binom{2j}{j} \binom{2k}{k} \frac{1}{x_1^{2j}} \frac{1}{x_2^{2k}} + \sum_{j,k=0}^{\infty} \frac{1}{j+k+1} \binom{2j}{j} \binom{2k}{k} \frac{1}{x_1^{2j+1}} \frac{1}{x_2^{2k+1}}.$$

Since

$$(4.5) \quad dx = \left(1 - \frac{1}{z^2}\right) dz,$$

we have

$$(4.6) \quad x \frac{d}{dx} = \frac{z + \frac{1}{z}}{1 - \frac{1}{z^2}} \frac{d}{dz} = \frac{z(z^2 + 1)}{z^2 - 1} \frac{d}{dz}.$$

To make the computation simpler, let us introduce

$$(4.7) \quad \xi_0(x) = \sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{x^{2m+1}}.$$

This will also be used in Section 7. In terms of z and t we have

$$(4.8) \quad \xi_0(x) = \frac{1}{2} \left(1 - x \frac{d}{dx}\right) \sum_{m=0}^{\infty} \frac{1}{m+1} \binom{2m}{m} \frac{1}{x^{2m+1}} \\ = \frac{1}{2} \left(1 - \frac{z(z^2 + 1)}{z^2 - 1} \frac{d}{dz}\right) z = -\frac{z}{z^2 - 1} = -\frac{t^2 - 1}{4t}.$$

Note that

$$-\left(x_1 \frac{d}{dx_1} + x_2 \frac{d}{dx_2}\right) \left(\frac{1}{4} \sum_{j,k=1}^{\infty} \frac{1}{j+k} \binom{2j}{j} \binom{2k}{k} \frac{1}{x_1^{2j}} \frac{1}{x_2^{2k}}\right)$$

$$\begin{aligned}
& + \sum_{j,k=0}^{\infty} \frac{1}{j+k+1} \binom{2j}{j} \binom{2k}{k} \frac{1}{x_1^{2j+1}} \frac{1}{x_2^{2k+1}} \\
& = \frac{1}{2} (x_1 \xi_0(x_1) - 1)(x_2 \xi_0(x_2) - 1) + 2\xi_0(x_1)\xi_0(x_2) \\
& \quad = 2z_1 z_2 \frac{1 + z_1 z_2}{(z_1^2 - 1)(z_2^2 - 1)} \\
& \quad = - \left(\frac{z_1(z_1^2 + 1)}{z_1^2 - 1} \frac{d}{dz_1} + \frac{z_2(z_2^2 + 1)}{z_2^2 - 1} \frac{d}{dz_2} \right) (-\log(1 - z_1 z_2)).
\end{aligned}$$

In other words, we have a partial differential equation

$$\left(x_1 \frac{d}{dx_1} + x_2 \frac{d}{dx_2} \right) (F_{0,2}^D(t_1, t_2) + \log(1 - z_1 z_2)) = 0$$

for a holomorphic function in x_1 and x_2 defined for $|x_1| \gg 2$ and $|x_2| \gg 2$. Since the first few terms of the Laurent expansions of $-\log(1 - z(x_1)z(x_2))$ using (3.7) agree with the first few terms of the sums of (4.4), we have the initial condition for the above differential equation. By the uniqueness of the solution to the Euler differential equation with the initial condition, we obtain (4.2). Equation (4.3) follows from differentiation of (4.2). \square

In terms of the t -coordinate of (4.1), the Galois conjugate of $t \in \Sigma$ under the projection $x : \Sigma \rightarrow \mathbb{C}$ is $-t$. Therefore, the recursion kernel for counting of dessins is given by

$$\begin{aligned}
(4.9) \quad K^D(t, t_1) & = \frac{1}{2} \frac{\int_t^{-t} W_{0,2}^D(\cdot, t_1)}{W_{0,1}^D(-t) - W_{0,1}^D(t)} = \frac{1}{2} \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{1}{\frac{t+1}{t-1} - \frac{t-1}{t+1}} \cdot \frac{1}{dx} \cdot dt_1 \\
& = -\frac{1}{64} \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \cdot \frac{1}{dt} \cdot dt_1.
\end{aligned}$$

One of the first two stable cases (2.4) gives us

$$\begin{aligned}
(4.10) \quad W_{1,1}^D(t_1) & = \frac{1}{2\pi i} \int_{\gamma} K^D(t, t_1) \left[W_{0,2}^D(t, -t) + \frac{dx \cdot dx_1}{(x-x_1)^2} \right] \\
& = -\frac{1}{2\pi i} \int_{\gamma} K^D(t, t_1) \frac{dt \cdot dt}{4t^2} = -\frac{1}{128} \frac{(t_1^2-1)^3}{t_1^4} dt_1,
\end{aligned}$$

where the integration contour γ consists of two concentric circles of a small radius and a large radius centered around $t = 0$, with the inner circle positively and the outer circle negatively oriented (Figure 4.1). The $(g, n) = (0, 3)$ case is given by

$$\begin{aligned}
(4.11) \quad W_{0,3}^D(t_1, t_2, t_3) & = \frac{1}{2\pi i} \int_{\gamma} \frac{W_{0,2}^D(t, t_1) W_{0,2}^D(t, t_2) W_{0,2}^D(t, t_3)}{dx(t) \cdot dy(t)} \\
& = -\frac{1}{16} \left[\frac{1}{2\pi i} \int_{\gamma} \frac{(t^2-1)^2(t-1)^2}{(t+t_1)^2(t+t_2)^2(t+t_3)^2} \cdot \frac{dt}{t} \right] dt_1 dt_2 dt_3 \\
& = -\frac{1}{16} \left(1 - \frac{1}{t_1^2 t_2^2 t_3^2} \right) dt_1 dt_2 dt_3.
\end{aligned}$$

Remark 4.2. The general formula (2.3) for $(g, n) = (0, 3)$ also gives the same answer. This is because $W_{0,2}^D$ acts as the Cauchy differentiation kernel.

$$\begin{aligned}
W_{0,3}^D(t_1, t_2, t_3) &= \frac{1}{2\pi i} \int_{\gamma} K^D(t, t_1) \left[W_{0,2}^D(t, t_2) W_{0,2}^D(-t, t_3) + W_{0,2}^D(t, t_3) W_{0,2}^D(-t, t_2) \right] \\
&= \frac{1}{64} \left[\frac{1}{2\pi i} \int_{\gamma} \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \left(\frac{1}{(t+t_2)^2(t-t_3)^2} + \frac{1}{(t+t_3)^2(t-t_2)^2} \right) dt \right] \\
&\quad \cdot dt_1 dt_2 dt_3 \\
&= \left[-\frac{1}{32} \frac{(t_1^2-1)^3}{t_1^2} \left(\frac{1}{(t_1+t_2)^2(t_1-t_3)^2} + \frac{1}{(t_1+t_3)^2(t_1-t_2)^2} \right) \right. \\
&\quad \left. - \frac{1}{16} \frac{\partial}{\partial t_2} \left(\frac{t_2}{t_2^2-t_1^2} \frac{(t_2^2-1)^3}{t_2^2} \frac{1}{(t_2+t_3)^2} \right) \right. \\
&\quad \left. - \frac{1}{16} \frac{\partial}{\partial t_3} \left(\frac{t_3}{t_3^2-t_1^2} \frac{(t_3^2-1)^3}{t_3^2} \frac{1}{(t_2+t_3)^2} \right) \right] dt_1 dt_2 dt_3 = -\frac{1}{16} \left(1 - \frac{1}{t_1^2 t_2^2 t_3^2} \right) dt_1 dt_2 dt_3.
\end{aligned}$$

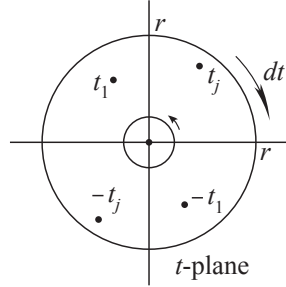


FIGURE 4.1. The integration contour γ . This contour encloses an annulus bounded by two concentric circles centered at the origin. The outer one has a large radius $r > \max_{j \in N} |t_j|$ and the negative orientation, and the inner one has an infinitesimally small radius with the positive orientation.

Theorem 4.3. *Let us define the Laplace transform of the number of Grothendieck's dessins by*

$$(4.12) \quad F_{g,n}^D(t_1, \dots, t_n) = \sum_{\mu \in \mathbb{Z}_+^n} D_{g,n}(\mu) e^{-(\mu_1 w_1 + \dots + \mu_n w_n)},$$

where the coordinate t_i is related to the Laplace conjugate coordinate w_j by

$$e^{w_j} = \frac{t_j + 1}{t_j - 1} + \frac{t_j - 1}{t_j + 1}.$$

Then the differential forms

$$(4.13) \quad W_{g,n}^D(t_1, \dots, t_n) = d_1 \cdots d_n F_{g,n}^D(t_1, \dots, t_n)$$

satisfy the Eynard-Orantin topological recursion

$$\begin{aligned}
(4.14) \quad W_{g,n}^D(t_1, \dots, t_n) &= -\frac{1}{64} \frac{1}{2\pi i} \int_{\gamma} \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \cdot \frac{1}{dt} \cdot dt_1 \\
&\times \left[\sum_{j=2}^n \left(W_{0,2}^D(t, t_j) W_{g,n-1}(-t, t_2, \dots, \widehat{t}_j, \dots, t_n) + W_{0,2}^D(-t, t_j) W_{g,n-1}(t, t_2, \dots, \widehat{t}_j, \dots, t_n) \right) \right]
\end{aligned}$$

$$+ W_{g-1, n+1}^D(t, -t, t_2, \dots, t_n) + \left. \sum_{\substack{\text{stable} \\ g_1+g_2=g \\ I \sqcup J = \{2, 3, \dots, n\}}} W_{g_1, |I|+1}^D(t, t_I) W_{g_2, |J|+1}^D(-t, t_J) \right].$$

The last sum is restricted to the stable geometries. In other words, the partition should satisfy $2g_1 - 1 + |I| > 0$ and $2g_2 - 1 + |J|$. The spectral curve Σ of the Eynard-Orantin recursion is given by

$$\begin{cases} x = z + \frac{1}{z} \\ y = -z \end{cases}$$

with the preferred coordinate t given by

$$t = \frac{z+1}{z-1}.$$

We give the proof of this theorem in the appendix.

The problem of counting dessins is closely related to the counting problem of the lattice points of the moduli space $\mathcal{M}_{g,n}$ of smooth n -pointed algebraic curves of genus g studied in [59, 60]. Let us briefly recall the combinatorial model for the moduli space $\mathcal{M}_{g,n}$ due to Thurston (see for example, [73]), Harer [36], Mumford [58], and Strebel [75], following [54, 55]. For a given ribbon graph Γ with $e = e(\Gamma)$ edges, the space of metric ribbon graphs is $\mathbb{R}_+^{e(\Gamma)} / \text{Aut}(\Gamma)$, where the automorphism group acts by permutations of edges (see [54, Section 1]). When we consider *ribbon graph automorphisms*, we restrict ourselves that $\text{Aut}(\Gamma)$ fixes each 2-cell of the cell-decomposition. We also require that every vertex of a ribbon graph has degree 3 or more. Using the canonical holomorphic coordinate system on a topological surface of [54, Section 4] corresponding to a metric ribbon graph, and the Strebel differentials [75], we have an isomorphism of topological orbifolds [36, 58]

$$(4.15) \quad \mathcal{M}_{g,n} \times \mathbb{R}_+^n \cong R_{g,n}$$

for (g, n) in the stable range. Here

$$R_{g,n} = \coprod_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{\mathbb{R}_+^{e(\Gamma)}}{\text{Aut}(\Gamma)}$$

is an orbifold consisting of metric ribbon graphs of a given topological type (g, n) . The gluing of orbi-cells is done by making the length of a non-loop edge tend to 0. The space $R_{g,n}$ is a smooth orbifold (see [54, Section 3] and [73]). We denote by $\pi : R_{g,n} \rightarrow \mathbb{R}_+^n$ the natural projection via (4.15), which is the assignment of the perimeter length of each boundary to a given metric ribbon graph.

Take a ribbon graph Γ . Since $\text{Aut}(\Gamma)$ fixes every boundary component of Γ , they are labeled by $[n] = \{1, 2, \dots, n\}$. For the moment let us give a label to each edge of Γ by an index set $[e] = \{1, 2, \dots, e\}$. The edge-face incidence matrix is defined by

$$(4.16) \quad \begin{aligned} A_\Gamma &= [a_{i\eta}]_{i \in [n], \eta \in [e]}; \\ a_{i\eta} &= \text{the number of times edge } \eta \text{ appears in face } i. \end{aligned}$$

Thus $a_{i\eta} = 0, 1$, or 2 , and the sum of the entries in each column is always 2. The Γ contribution of the space $\pi^{-1}(\mu_1, \dots, \mu_n) = R_{g,n}(\mu)$ of metric ribbon graphs with a prescribed perimeter $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}_+^n$ is the orbifold polytope

$$\frac{\{\mathbf{x} \in \mathbb{R}_+^e \mid A_\Gamma \mathbf{x} = \mu\}}{\text{Aut}(\Gamma)},$$

where $\mathbf{x} = (\ell_1, \dots, \ell_e)$ is the collection of edge lengths of the metric ribbon graph Γ . We have

$$(4.17) \quad \sum_{i \in [n]} \mu_i = \sum_{i \in [n]} \sum_{\eta \in [e]} a_{i\eta} \ell_\eta = 2 \sum_{\eta \in [e]} \ell_\eta.$$

Now let $\mu \in \mathbb{Z}_+^n$ be a vector consisting of positive integers. The lattice point counting function we consider is defined by

$$(4.18) \quad N_{g,n}(\mu) = \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{|\{\mathbf{x} \in \mathbb{Z}_+^n \mid A_\Gamma \mathbf{x} = \mu\}|}{|\text{Aut}(\Gamma)|}$$

for (g, n) in the stable range ([10, 55, 59, 60]).

To find the spectral curve for lattice point counting, we need to identify the unstable moduli $\mathcal{M}_{0,1}$ and the ribbon graph space $R_{0,1}$. We recall that the orbifold isomorphism (4.15) holds for (g, n) in the stable range by *defining* $R_{g,n}$ as the space of metric ribbon graphs of type (g, n) *without* vertices of degrees 1 and 2. For $(g, n) = (0, 1)$, there are no ribbon graphs satisfying these conditions. Let v_j denote the number of degree j vertices in a ribbon graph Γ of type (g, n) . Then we have

$$\sum_{j \geq 1} j v_j = 2e, \quad \sum_{j \geq 1} v_j = v,$$

where v is the total number of vertices of Γ . Hence

$$(4.19) \quad 2(2g - 2 + n) = 2e - 2v = \sum_{j \geq 1} (j - 2)v_j = -v_1 + \sum_{j \geq 3} (j - 2)v_j.$$

It follows that the number of degree 1 vertices v_1 is positive when $(g, n) = (0, 1)$. Thus we conclude that *there is no spectral curve for this counting problem*.

Still we can consider the Laplace transform of the number (4.18) of lattice points of the moduli space $\mathcal{M}_{g,n}$ with a prescribed perimeter length. We define for every stable (g, n)

$$(4.20) \quad F_{g,n}^L(t_1, \dots, t_n) = \sum_{\mu \in \mathbb{Z}_+^n} N_{g,n}(\mu) \prod_{i=1}^n \frac{1}{z_i^{\mu_i}},$$

and the Eynard-Orantin differential forms by

$$(4.21) \quad W_{g,n}^L(t_1, \dots, t_n) = d_1 \cdots d_n F_{g,n}^L(t_1, \dots, t_n).$$

The following result is proved in [10], with inspiration from [60].

Theorem 4.4 ([10]). *The differential forms $W_{g,n}^L(t_1, \dots, t_n)$ satisfy the Eynard-Orantin topological recursion with respect to the same spectral curve (3.13) and the recursion kernel (4.9), starting with exactly the same first two stable cases*

$$(4.22) \quad W_{1,1}^L(t_1) = -\frac{1}{128} \frac{(t_1^2 - 1)^3}{t_1^4} dt_1,$$

and

$$(4.23) \quad W_{0,3}^L(t_1, t_2, t_3) = -\frac{1}{16} \left(1 - \frac{1}{t_1^2 t_2^2 t_3^2} \right) dt_1 dt_2 dt_3.$$

Remark 4.5. It is somewhat surprising, because the spectral curve (3.13) has nothing to do with the lattice point counting problem. As we have mentioned, the $(g, n) = (0, 1)$ and $(0, 2)$ considerations for this problem do not produce the spectral curve.

Corollary 4.6. *For every (g, n) with $2g - 2 + n > 0$, we have the identity*

$$(4.24) \quad W_{g,n}^D(t_1, \dots, t_n) = W_{g,n}^L(t_1, \dots, t_n).$$

The differential form $W_{g,n}^D(t_1, \dots, t_n)$ is a Laurent polynomial in t_1^2, \dots, t_n^2 of degree $2(3g - 3 + n)$, with a reciprocity property

$$(4.25) \quad W_{g,n}^D(1/t_1, \dots, 1/t_n) = (-1)^n t_1^2 \cdots t_n^2 W_{g,n}^D(t_1, \dots, t_n).$$

The numbers of dessins can be expressed in terms of the number of lattice points:

$$(4.26) \quad D_{g,n}(\mu_1, \dots, \mu_n) = \sum_{\ell_1 > \frac{\mu_1}{2}} \cdots \sum_{\ell_n > \frac{\mu_n}{2}} \prod_{i=1}^n \frac{2\ell_i - \mu_i}{\mu_i} \binom{\mu_i}{\ell_i} N_{g,n}(2\ell_1 - \mu_1, \dots, 2\ell_n - \mu_n).$$

Remark 4.7. The relation (4.26) appears in [62, Section 2.1] for an abstract setting.

Proof. The Eynard-Orantin topological recursion uniquely determines the differential forms for all (g, n) . Since $W_{1,1}^D(t) = W_{1,1}^L(t)$ and $W_{0,3}^D(t_1, t_2, t_3) = W_{0,3}^L(t_1, t_2, t_3)$, we conclude that $W_{g,n}^D(t_1, \dots, t_n) = W_{g,n}^L(t_1, \dots, t_n)$ for $2g - 2 + n > 0$.

By induction on $2g - 2 + n$ we can show that $W_{g,n}^D(t_1, \dots, t_n)$ is a Laurent polynomial in t_1^2, \dots, t_n^2 . The statement is true for the initial cases (4.10) and (4.11). The integral transformation formula (4.14) is a residue calculation at $t = 0$ and $t = \infty$. By the induction hypothesis, the right-hand side of (4.14) becomes

$$\begin{aligned} & -\frac{1}{64} \frac{1}{2\pi i} \int_{\gamma} \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \cdot \frac{1}{dt} \cdot dt_1 \\ & \times \left[\sum_{j=2}^n \left(W_{0,2}^D(t, t_j) W_{g,n-1}(-t, t_2, \dots, \hat{t}_j, \dots, t_n) + W_{0,2}^D(-t, t_j) W_{g,n-1}(t, t_2, \dots, \hat{t}_j, \dots, t_n) \right) \right. \\ & \quad \left. + W_{g-1,n+1}^D(t, -t, t_2, \dots, t_n) + \sum_{\substack{\text{stable} \\ g_1+g_2=g \\ I \sqcup J = \{2,3,\dots,n\}}} W_{g_1,|I|+1}^D(t, t_I) W_{g_2,|J|+1}^D(-t, t_J) \right] \\ & = \frac{1}{32} \frac{1}{2\pi i} \int_{\gamma} \frac{(t^2-1)^3}{t^2-t_1^2} \frac{1}{t} \cdot \frac{1}{dt} \cdot dt_1 \left[\sum_{j=2}^n \frac{2(t^2+t_j^2)}{(t^2-t_j^2)^2} W_{g,n-1}(t, t_2, \dots, \hat{t}_j, \dots, t_n) dt \cdot dt_j \right. \\ & \quad \left. + W_{g-1,n+1}^D(t, t, t_2, \dots, t_n) + \sum_{\substack{\text{stable} \\ g_1+g_2=g \\ I \sqcup J = \{2,3,\dots,n\}}} W_{g_1,|I|+1}^D(t, t_I) W_{g_2,|J|+1}^D(t, t_J) \right]. \end{aligned}$$

Clearly the residues at $t = 0$ and $t = \infty$ are Laurent polynomials in t_1^2, \dots, t_n^2 .

Because of (4.24), we have

$$(4.27) \quad \sum_{\mu \in \mathbb{Z}_+^n} D_{g,n}(\mu) \prod_{i=1}^n d\left(\frac{1}{x_i^{\mu_i}}\right) = \sum_{\nu \in \mathbb{Z}_+^n} N_{g,n}(\nu) \prod_{i=1}^n d\left(\frac{1}{z_i^{\nu_i}}\right) = (-1)^n \sum_{\nu \in \mathbb{Z}_+^n} N_{g,n}(\nu) \prod_{i=1}^n dz_i^{\nu_i},$$

where $x_i = z_i + 1/z_i$. The Galois conjugation $t \rightarrow -t$ corresponds to $z \rightarrow 1/z$. Since

$$W_{g,n}^N(t_1, \dots, t_n) = (-1)^n W_{g,n}^N(-t_1, \dots, -t_n),$$

the second equality of (4.27) follows. Take the residue of the left-hand side of (4.27) at $x_i = \infty$ for $i = 1, \dots, n$. On the right-hand side we take the residue at $z_i = 0$ for every i .

Then for every $(\mu_1, \dots, \mu_n) \in \mathbb{Z}_+^n$ we have
(4.28)

$$D_{g,n}(\mu_1, \dots, \mu_n) \mu_1 \cdots \mu_n = \left(\frac{1}{2\pi i} \right)^n \int_{|z_1|=\epsilon} \cdots \int_{|z_n|=\epsilon} x_1^{\mu_1} \cdots x_n^{\mu_n} \sum_{\nu \in \mathbb{Z}_+^n} N_{g,n}(\nu) \prod_{i=1}^n dz_i^{\nu_i}.$$

Since

$$\left(z_i + \frac{1}{z_i} \right)^{\mu_i} = \sum_{\ell_i=0}^{\mu_i} \binom{\mu_i}{\ell_i} z_i^{\mu_i - 2\ell_i},$$

the residue of (4.28) comes from the term $\mu_i - 2\ell_i + \nu_i = 0$, and we have

$$\begin{aligned} & D_{g,n}(\mu_1, \dots, \mu_n) \mu_1 \cdots \mu_n \\ &= \sum_{\ell_1 > \mu_1/2} \cdots \sum_{\ell_n > \mu_n/2} \prod_{i=1}^n (2\ell_i - \mu_i) \binom{\mu_i}{\ell_i} N_{g,n}(2\ell_1 - \mu_1, \dots, 2\ell_n - \mu_n). \end{aligned}$$

The reciprocity relation, and the degree of the Laurent polynomial, is the consequence of the following, which was established in [55].

Theorem 4.8 ([55]). *The functions $F_{g,n}^L(t_1, \dots, t_n)$ of (4.20) for the stable range $2g - 2 + n > 0$ are uniquely determined by the following differential recursion formula from the initial values $F_{0,3}^L(t_1, t_2, t_3)$ and $F_{1,1}^L(t_1)$.*

$$\begin{aligned} (4.29) \quad & F_{g,n}^L(t_1, \dots, t_n) \\ &= -\frac{1}{16} \int_{-1}^{t_1} \left[\sum_{j=2}^n \frac{t_j}{t^2 - t_j^2} \left(\frac{(t^2 - 1)^3}{t^2} \frac{\partial}{\partial t} F_{g,n-1}^L(t, t_{[n] \setminus \{1,j\}}) - \frac{(t_j^2 - 1)^3}{t_j^2} \frac{\partial}{\partial t_j} F_{g,n-1}^L(t_{[n] \setminus \{1\}}) \right) \right. \\ &\quad \left. + \sum_{j=2}^n \frac{(t^2 - 1)^2}{t^2} \frac{\partial}{\partial t} F_{g,n-1}^L(t, t_{[n] \setminus \{1,j\}}) \right. \\ &\quad \left. + \frac{1}{2} \frac{(t^2 - 1)^3}{t^2} \frac{\partial^2}{\partial u_1 \partial u_2} \left(F_{g-1,n+1}^L(u_1, u_2, t_{[n] \setminus \{1\}}) \right. \right. \\ &\quad \left. \left. + \sum_{\substack{\text{stable} \\ g_1+g_2=g \\ I \sqcup J = [n] \setminus \{1\}}} F_{g_1,|I|+1}^L(u_1, t_I) F_{g_2,|J|+1}^L(u_2, t_J) \right) \Big|_{u_1=u_2=t} \right] dt. \end{aligned}$$

Here $[n] = \{1, 2, \dots, n\}$ is an index set, and the last sum is taken over all partitions $g_1 + g_2 = g$ and set partitions $I \sqcup J = [n] \setminus \{1\}$ subject to the stability conditions $2g_1 - 1 + |I| > 0$ and $2g_2 - 1 + |J| > 0$. The initial values are given by

$$(4.30) \quad F_{1,1}^L(t_1) = -\frac{1}{384} \frac{(t+1)^4}{t^2} \left(t - 4 + \frac{1}{t} \right)$$

and

$$(4.31) \quad F_{0,3}^L(t_1, t_2, t_3) = -\frac{1}{16} (t_1 + 1)(t_2 + 1)(t_3 + 1) \left(1 + \frac{1}{t_1 t_2 t_3} \right).$$

In the stable range $F_{g,n}^L(t_1, \dots, t_n)$ is a Laurent polynomial of degree $3(2n - 2 + n)$ and satisfies the reciprocity relation

$$(4.32) \quad F_{g,n}^L(1/t_1, \dots, 1/t_n) = F_{g,n}^L(t_1, \dots, t_n).$$

The leading terms of $F_{g,n}^L(t_1, \dots, t_n)$ form a homogeneous polynomial of degree $3(2g-2+n)$, and is given by

$$(4.33) \quad F_{g,n}^K(t_1, \dots, t_n) \stackrel{\text{def}}{=} \frac{(-1)^n}{2^{2g-2+n}} \sum_{\substack{d_1+\dots+d_n \\ =3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{j=1}^n (2d_j - 1)!! \left(\frac{t_j}{2}\right)^{2d_j+1},$$

where

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$$

is the ψ -class intersection number (see Section 6 for more detail about intersection numbers). The special value at $t_i = 1$ gives

$$(4.34) \quad F_{g,n}^L(1, 1, \dots, 1) = (-1)^n \chi(\mathcal{M}_{g,n}).$$

This completes the proof of Corollary 4.6. \square

5. THE ψ -CLASS INTERACTION NUMBERS ON $\overline{\mathcal{M}}_{g,n}$

The crucial discovery of Kontsevich [46] is the equality between the intersection numbers on the compact moduli space $\overline{\mathcal{M}}_{g,n}$ and the Euclidean volume of the moduli space $\mathcal{M}_{g,n}$ of smooth curves using the isomorphism (4.15). The Feynman diagram expansion of the Kontsevich matrix integral relates the Euclidean volume with a τ -function of the KdV equations. The Eynard-Orantin recursion for the ψ -class intersection numbers is precisely the Dijkgraaf-Verlinde-Verlinde formula [15] of the intersection numbers. In this section we identify the spectral curve and the recursion kernel for the ψ -class intersection numbers.

As we have noted, the derivative of the recursion formula (4.29) is *not* the Eynard-Orantin recursion because the spectral curve is not defined by the unstable geometries. Indeed, we have $dF_{0,1}^L \equiv 0$. However, when we associate the number of lattice points with the ψ -class intersection numbers on $\overline{\mathcal{M}}_{g,n}$, the unstable geometries do make sense.

Let us recall a computation in [55, Section 4].

$$(5.1) \quad \begin{aligned} \sum_{\mu \in \mathbb{Z}_+^n} N_{g,n}(\mu) e^{-\langle \mu, w \rangle} &= \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \sum_{\mu \in \mathbb{Z}_+^n} \frac{1}{|\text{Aut}(\Gamma)|} |\{\mathbf{x} \in \mathbb{Z}_+^{e(\Gamma)} \mid A_\Gamma \mathbf{x} = \mu\}| e^{-\langle \mu, w \rangle} \\ &= \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{1}{|\text{Aut}(\Gamma)|} \sum_{\mathbf{x} \in \mathbb{Z}_+^{e(\Gamma)}} e^{-\langle A_\Gamma \mathbf{x}, w \rangle} \\ &= \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{\substack{\eta \text{ edge} \\ \text{of } \Gamma}} \sum_{\ell_\eta=1}^{\infty} e^{-\langle a_\eta, w \rangle \ell_\eta} \\ &= \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{\substack{\eta \text{ edge} \\ \text{of } \Gamma}} \frac{e^{-\langle a_\eta, w \rangle}}{1 - e^{-\langle a_\eta, w \rangle}}, \end{aligned}$$

where A_Γ is the incidence matrix of (4.16), a_η is the η -th column of A_Γ , and $\langle \mu, w \rangle = \mu_1 w_1 + \cdots + \mu_n w_n$. By comparing (4.20) and (5.1), we see that we are substituting $e^{w_i} = z_i$

in this computation. Therefore, we obtain

$$(5.2) \quad F_{g,n}^L(t_1, \dots, t_n) = \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{\substack{\eta \text{ edge} \\ \text{of } \Gamma}} \frac{1}{\prod_{i=1}^n z_i^{a_{i\eta}} - 1}.$$

Thus the series (4.20) in z_i converges for $|z_i| > 1$. Since $z_i = \frac{t_i+1}{t_i-1}$, the $t_i \rightarrow \infty$ limit picks up the limit of (4.20) as $z_i \rightarrow 1$, and hence the information of $N_{g,n}(\mu)$ as $\mu_i \rightarrow \infty$. Since the orbifold isomorphism (4.15) is scale invariant under the action of \mathbb{R}_+ , making the perimeter length μ large is the same as making the mesh small in the lattice point counting. Hence at the limit we obtain the Euclidean volume of $\mathcal{M}_{g,n}$ considered by Kontsevich in [46]. This is why we expect that (4.33) holds. Let us now consider the limit of the spectral curve (3.13) as $t \rightarrow \infty$. First we have

$$\begin{aligned} x &= z + \frac{1}{z} = 2 + \frac{4}{t^2 - 1} \\ y &= -z = -1 - \frac{2}{t-1}. \end{aligned}$$

Ignoring the constant shifts of x and y , we obtain for a large t

$$(5.3) \quad \begin{cases} x = \frac{4}{t^2} \\ y = -\frac{2}{t}. \end{cases}$$

Hence the spectral curve is given by the equation $x = y^2$. We use t as the preferred coordinate.

We now compare the Eynard-Orantin recursion with respect to this spectral curve and the Witten-Kontsevich theory. We use (4.33) and define

$$(5.4) \quad \begin{aligned} W_{g,n}^K(t_1, \dots, t_n) &= d_1 \cdots d_n F_{g,n}^K(t_1, \dots, t_n) \\ &= \frac{(-1)^n}{2^{2g-2+n}} \sum_{\substack{d_1 + \cdots + d_n \\ = 3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{j=1}^n (2d_j + 1)!! \left(\frac{t_j}{2}\right)^{2d_j} d\left(\frac{t_j}{2}\right) \\ &= \frac{(-1)^n}{16^{2g-2+n}} w_{g,n}^K(t_1, \dots, t_n) dt_1 \cdots dt_n, \end{aligned}$$

where $w_{g,n}^K(t_1, \dots, t_n)$ is the coefficient of the Eynard-Orantin differential form normalized by the constant factor $\frac{(-1)^n}{16^{2g-2+n}}$. Note that $w_{g,n}^K(t_1, \dots, t_n)$ is a polynomial in t_i^2 's with positive rational coefficients for (g, n) in the stable range. For $(g, n) = (0, 1)$ and $(0, 2)$, we have

$$(5.5) \quad \langle \tau_k \rangle_{0,1} = \delta_{k+2,0}$$

$$(5.6) \quad \langle \tau_{k_1} \tau_{k_2} \rangle_{0,2} = (-1)^{k_1}, \quad k_1 + k_2 = -1.$$

Therefore,

$$(5.7) \quad W_{0,1}^K(t) = \frac{-1}{16^{-1}} \langle \tau_{-2} \rangle (-3)!! t^{-4} dt = \frac{16}{t^4} dt = y dx,$$

in agreement with the spectral curve $x = y^2$ (5.3). Similarly, we have

$$(5.8) \quad F_{0,2}^K(t_1, t_2) = \sum_{d=0}^{\infty} (-1)^d (2d-1)!! (-2d-3)!! \left(\frac{t_1}{2}\right)^{2d+1} \left(\frac{t_2}{2}\right)^{-2d-1}$$

$$= - \sum_{d=0}^{\infty} \frac{1}{2d+1} \left(\frac{t_1}{t_2} \right)^{2d+1} = \log \left(1 - \frac{t_1}{t_2} \right) - \frac{1}{2} \log \left(1 - \frac{t_1^2}{t_2^2} \right),$$

and hence

$$(5.9) \quad W_{0,2}^K(t_1, t_2) = \frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2} - \frac{1}{2} \frac{dx_1 \cdot dx_2}{(x_1 - x_2)^2}.$$

As a consequence, the recursion kernel is given by

$$(5.10) \quad K^K(t, t_1) = -\frac{1}{2} \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{t^4}{32} \frac{1}{dt} dt_1,$$

since $\frac{dx_1 \cdot dx_2}{(x_1 - x_2)^2}$ does not contribute to the kernel. The Eynard-Orantin recursion for the Euclidean volume then becomes

$$(5.11) \quad \begin{aligned} W_{g,n}^K(t_1, \dots, t_n) &= -\frac{1}{2\pi i} \int_{\gamma_\infty} \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{t^4}{64} \frac{1}{dt} dt_1 \left[W_{g-1, n+1}^K(t, -t, t_2, \dots, t_n) \right. \\ &\quad + \sum_{j=2}^n \left(\frac{dt \cdot dt_j}{(t-t_j)^2} W_{g, n-1}^K(-t, t_2, \dots, \widehat{t}_j, \dots, t_n) \right. \\ &\quad \left. \left. - \frac{dt \cdot dt_j}{(t+t_j)^2} W_{g, n-1}^K(t, t_2, \dots, \widehat{t}_j, \dots, t_n) \right) \right. \\ &\quad \left. + \sum_{\substack{\text{stable} \\ g_1+g_2=g \\ I \sqcup J = \{2, \dots, n\}}} W_{g_1, |I|+1}^K(t, t_I) W_{g_2, |J|+1}^K(-t, t_J) \right], \end{aligned}$$

where the integral is taken with respect to a large negatively oriented circle γ_∞ that encloses any of $\pm t_1, \dots, \pm t_n$. This is the larger circle of Figure 4.1. Here again $\frac{dx_1 \cdot dx_2}{(x_1 - x_2)^2}$ does not contribute in the formula. Since the coefficients $w_{g,n}^K(t_1, \dots, t_n)$ in the stable range are polynomials, the poles of the integrand of (5.11) in the integration contour are at $t = \pm t_i$'s. Therefore, we can perform the integral in terms of the residue calculus at poles $t = \pm t_i$. First let us get rid of the factor $1/16^{2g-2+n}$ from (5.11). Since the recursion is an induction on $2g-2+n$, we have an overall factor 16 adjustment on the right-hand side. The integration contour is negatively oriented, so the residue calculation at $t = \pm t_i$ receives universally the negative sign. This sign is exactly cancelled by the choice of the sign of $w_{g,n}^K$ in (5.4). Thus the result of residue evaluation of (5.11) is

$$(5.12) \quad \begin{aligned} w_{g,n}^K(t_1, \dots, t_n) &= \frac{1}{2} t_1^4 w_{g-1, n+1}^K(t_1, t_1, t_2, \dots, t_n) \\ &\quad + \frac{1}{2} t_1^4 \sum_{\substack{\text{stable} \\ g_1+g_2=g \\ I \sqcup J = \{2, \dots, n\}}} w_{g_1, |I|+1}^K(t_1, t_I) w_{g_2, |J|+1}^K(t_1, t_J) \\ &\quad + t_1^4 \sum_{j=2}^n \frac{t_1^2 + t_j^2}{(t_1^2 - t_j^2)^2} w_{g, n-1}^K(t_1, \dots, \widehat{t}_j, \dots, t_n) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{j=2}^n \left(\frac{\partial}{\partial t} \Big|_{t=t_j} + \frac{\partial}{\partial t} \Big|_{t=-t_j} \right) \left(\frac{1}{t^2 - t_1^2} t^5 w_{g,n-1}^K(t, t_2, \dots, \widehat{t_j}, \dots, t_n) \right) \\
& = \frac{1}{2} t_1^4 \left[w_{g-1,n+1}^K(t_1, t_1, t_2, \dots, t_n) + \sum_{\substack{\text{stable} \\ g_1+g_2=g \\ I \sqcup J = \{2, \dots, n\}}} w_{g_1, |I|+1}^K(t_1, t_I) w_{g_2, |J|+1}^K(t_1, t_J) \right] \\
& \quad + \sum_{j=2}^n \frac{\partial}{\partial t_j} \left[\frac{t_j}{t_1^2 - t_j^2} \left(t_1^4 w_{g,n-1}^K(t_{[n] \setminus \{j\}}) - t_j^4 w_{g,n-1}^K(t_{[n] \setminus \{1\}}) \right) \right].
\end{aligned}$$

This is the same as [10, Theorem 5.2].

Let us adopt the normalized notation

$$(5.13) \quad \langle \sigma_{d_1} \cdots \sigma_{d_n} \rangle_{g,n} = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^n (2d_i + 1)!!$$

to make the formula shorter. Then

$$(5.14) \quad w_{g,n}^K(t_1, \dots, t_n) = \sum_{d_1, \dots, d_n} \langle \sigma_{d_1} \cdots \sigma_{d_n} \rangle_{g,n} \prod_{j=1}^n t_j^{2d_j}.$$

The DVV formula [15] for the Virasoro constraint condition on the ψ -class intersection numbers on $\overline{\mathcal{M}}_{g,n}$ reads

$$\begin{aligned}
(5.15) \quad \langle \sigma_k \prod_{i=2}^n \sigma_{d_i} \rangle_{g,n} &= \frac{1}{2} \sum_{a+b=k-2} \langle \sigma_a \sigma_b \prod_{i=2}^n \sigma_{d_i} \rangle_{g-1, n+1} \\
& + \frac{1}{2} \sum_{a+b=k-2} \sum_{\substack{\text{stable} \\ g_1+g_2=g \\ I \sqcup J = \{2, \dots, n\}}} \langle \sigma_a \prod_{i \in I} \sigma_{d_i} \rangle_{g_1, |I|+1} \cdot \langle \sigma_b \prod_{j \in J} \sigma_{d_j} \rangle_{g_2, |J|+1} \\
& \quad + \sum_{j=2}^n (2d_j + 1) \langle \sigma_{k+d_j-1} \prod_{i \neq 1, j} \sigma_{d_i} \rangle_{g, n-1}.
\end{aligned}$$

We thus recover the discovery of [25]:

Theorem 5.1. *The Eynard-Orantin recursion formula for the spectral curve $x = y^2$ is the Dijkgraaf-Verlinde-Verlinde formula [15] for the intersection numbers $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$ on the moduli space $\overline{\mathcal{M}}_{g,n}$ of pointed stable curves.*

Proof. We extract the coefficient of

$$(5.16) \quad t_1^{2k} \prod_{j=2}^n t_j^{2d_j}$$

in (5.12) and compare the result with (5.15). It is obvious that the fifth line of (5.12) produces the first and second lines of (5.15).

To compare the last lines of (5.12) and (5.15), we consider the case $|t_j| < |t_1|$ for all $j \geq 2$ in (5.12). We then have the expansion

$$\frac{1}{t_1^2 - t_j^2} = \frac{1}{t_1^2} \frac{1}{1 - \frac{t_j^2}{t_1^2}} = \frac{1}{t_1^2} \sum_{m=0}^{\infty} \left(\frac{t_j^2}{t_1^2} \right)^m.$$

The (5.16)-term of the last line of (5.12) has two contributions. The first one comes from

$$\frac{\partial}{\partial t_j} \left(t_1^2 t_j \sum_{m=0}^{\infty} \left(\frac{t_j^2}{t_1^2} \right)^m w_{g,n-1}^K(t_1, t_2, \dots, \hat{t}_j, \dots, t_n) \right).$$

Since $w_{g,n-1}^K(t_1, t_2, \dots, \hat{t}_j, \dots, t_n)$ does not contain t_j , we set $m = d_j$ to produce the right power $2d_j$ of t_j . The power of t_1 has to be $2k$. Thus from $w_{g,n-1}^K$ we take the term of $t_1^{2k+2d_j-2}$, whose coefficient is $\langle \sigma_{k+d_j-1} \prod_{i \neq 1, j} \sigma_{d_i} \rangle$. The total contribution from the first kind comes from the differentiation, which gives $2m + 1 = 2d_j + 1$.

The second possible contribution for the (5.16)-term may come from

$$-\frac{\partial}{\partial t_j} \left(\frac{t_j^5}{t_1^2} \sum_{m=0}^{\infty} \left(\frac{t_j^2}{t_1^2} \right)^m w_{g,n-1}^K(t_2, \dots, t_n) \right).$$

However, this term does not produce t_1^{2k} , and hence does not contribute to the (5.16)-term. This completes the proof of Theorem 5.1. \square

6. SINGLE HURWITZ NUMBERS

What is the mirror dual of the number of trees? The answer we wish to present in this section is that it is the *Lambert curve*. This analytic curve serves as the spectral curve for the Hurwitz counting problem, and comes up from the the unstable geometries $(g, n) = (0, 1)$ and $(0, 2)$ via Laplace transform.

A *Hurwitz cover* is a holomorphic mapping $f : C \rightarrow \mathbb{P}^1$ from a connected nonsingular projective algebraic curve C of genus g to the projective line \mathbb{P}^1 with only simple ramifications except for $\infty \in \mathbb{P}^1$. Such a cover is further refined by specifying its *profile*, which is a partition $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_n > 0)$ of the degree of the covering $d = |\mu| = \mu_1 + \dots + \mu_n$. The length $\ell(\mu) = n$ of this partition is the number of points in the inverse image $f^{-1}(\infty) = \{p_1, \dots, p_n\}$ of ∞ . Each part μ_i gives a local description of the map f , which is given by $u \mapsto u^{-\mu_i}$ in terms of a local coordinate u of C around p_i . The number $h_{g,\mu}$ of the topological types of Hurwitz covers of a given genus g and a profile μ , counted with the weight factor $1/|\text{Aut} f|$, is the *single Hurwitz number* we shall deal with in this section.

The deformations of a Hurwitz cover f are obtained by moving the branch points (i.e., the image of the ramification points) on $\mathbb{P}^1 \setminus \{\infty\}$. Thus $h_{g,\mu}$ counts the number of Hurwitz covers with prescribed (i.e., fixed) and *labeled* branch points. On the other hand, the preimages of ∞ on C are labeled only by the parts of μ . Therefore, a more natural count of Hurwitz cover is

$$(6.1) \quad H_g(\mu) = \frac{|\text{Aut}(\mu)|}{(2g - 2 + n + |\mu|)!} \cdot h_{g,\mu}.$$

Here,

$$(6.2) \quad r = r(g, \mu) \stackrel{\text{def}}{=} 2g - 2 + n + |\mu|$$

is the number of simple ramification points of f by the Riemann-Hurwitz formula, and $\text{Aut}(\mu)$ is the group of permutations of equal parts of the partition μ .

One reason that explains why single Hurwitz numbers are interesting is a remarkable formula due to Ekedahl, Lando, Shapiro and Vainshtein [20, 33, 48, 65] that relates Hurwitz numbers and Gromov-Witten invariants. For genus $g \geq 0$ and a partition μ of length $\ell(\mu) = n$ subject to the stability condition $2g - 2 + n > 0$, the ELSV formula states that

$$(6.3) \quad H_g(\mu) = \prod_{i=1}^n \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \\ = \sum_{j=0}^g (-1)^j \sum_{k_1, \dots, k_n \geq 0} \langle \tau_{k_1} \cdots \tau_{k_n} c_j(\mathbb{E}) \rangle \prod_{i=1}^n \frac{\mu_i^{\mu_i + k_i}}{\mu_i!},$$

where $\overline{\mathcal{M}}_{g,n}$ is the Deligne-Mumford moduli stack of stable algebraic curves of genus g with n distinct smooth marked points, $\Lambda_g^\vee(1) = 1 - c_1(\mathbb{E}) + \cdots + (-1)^g c_g(\mathbb{E})$ is the alternating sum of the Chern classes of the Hodge bundle \mathbb{E} on $\overline{\mathcal{M}}_{g,n}$, ψ_i is the i -th tautological cotangent class, and

$$(6.4) \quad \langle \tau_{k_1} \cdots \tau_{k_n} c_j(\mathbb{E}) \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} c_j(\mathbb{E})$$

is the linear Hodge integral, which is 0 unless $k_1 + \cdots + k_n + j = 3g - 3 + n$.

The Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}$ is defined as the moduli space of *stable* curves satisfying the stability condition $2 - 2g - n < 0$. However, single Hurwitz numbers are well defined for *unstable* geometries $(g, n) = (0, 1)$ and $(0, 2)$, and their values are

$$(6.5) \quad H_0((d)) = \frac{d^{d-3}}{(d-1)!} = \frac{d^{d-2}}{d!} \quad \text{and} \quad H_0((\mu_1, \mu_2)) = \frac{1}{\mu_1 + \mu_2} \cdot \frac{\mu_1^{\mu_1}}{\mu_1!} \cdot \frac{\mu_2^{\mu_2}}{\mu_2!}.$$

The ELSV formula remains valid for unstable cases by *defining*

$$(6.6) \quad \int_{\overline{\mathcal{M}}_{0,1}} \frac{\Lambda_0^\vee(1)}{1 - d\psi} = \frac{1}{d^2},$$

$$(6.7) \quad \int_{\overline{\mathcal{M}}_{0,2}} \frac{\Lambda_0^\vee(1)}{(1 - \mu_1 \psi_1)(1 - \mu_2 \psi_2)} = \frac{1}{\mu_1 + \mu_2}.$$

Let us examine the $(g, n) = (0, 1)$ case. We wish to count the number of Hurwitz covers $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree d with profile $\mu = (d)$. If $d = 2$, then $f(u) = u^2$ is the only map, since $r = 1$ and the two ramification points can be placed at $u = 0$ and $u = \infty$. The automorphism of this map is $\mathbb{Z}/2\mathbb{Z}$. We now consider the case when $d \geq 3$. First we label all branch points. One is ∞ , so let us place all others, the images of simple ramification points, at the r -th roots of unity. Here $r = d - 1$. We label these points with indices $[r] = \{1, 2, \dots, r\}$. Connect each r -th root of unity with the origin by a straight line (see Figure 6.1). Let $*$ denote this star-like shape, which has one vertex at the center and r half-edges. Then the inverse image $f^{-1}(*)$ is a tree-like shape with d vertices and rd half-edges. Here we call each inverse image of 0 a vertex of $f^{-1}(*)$. If f is simply ramified at p , then two half-edges are connected at p and form a real edge that is incident to two vertices. Since $f(p)$ is one of the r -th root of unity, we give the same label to p . Thus all simple ramification points are labeled with the index set $[r]$. Now we remove all half-edges from $f^{-1}(*)$ that are not made into an edge, and denote it by T . It is a tree on \mathbb{P}^1 that has d vertices and $r = d - 1$ edges. Note that except for the case $d = 2$, the edge labeling gives a labeling of vertices. For example, if a vertex x is incident to edges $i_1 < i_2 < \cdots < i_k$, then x is labeled by $i_1 i_2 \cdots i_k$.

Conversely, suppose we are given a tree with d labeled vertices by the index set $[d] = \{1, 2, \dots, d\}$ and $r = d - 1$ edges. At each vertex we can give a cyclic order to incident edges by aligning them in the increasing order of the labels of the other ends of the edges. Thus the tree becomes a *ribbon graph* (see Section 3), and hence it can be placed on \mathbb{P}^1 . Then by choosing the midpoint of each edge as a simple ramification point and each vertex as a

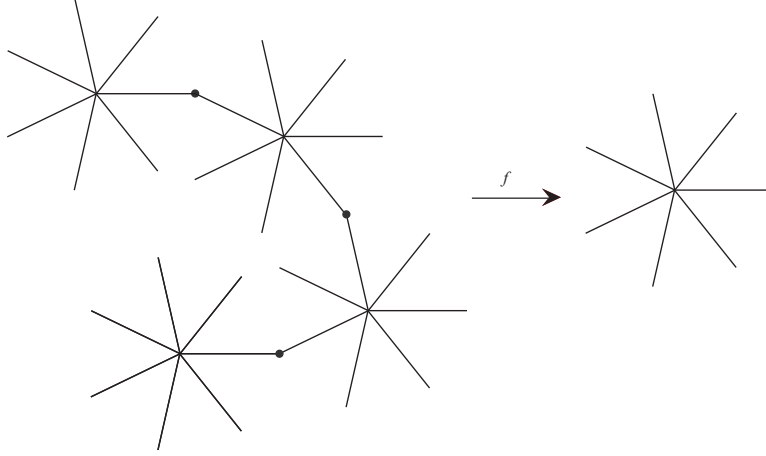


FIGURE 6.1. Counting the genus 0 single Hurwitz numbers with the total ramification at ∞ .

zero of f , we can construct a Hurwitz cover. Recall that the number of trees with d labeled vertices is d^{d-2} . Therefore,

$$H_0((d)) = \frac{d^{d-2}}{d!}$$

is the number of trees with d *unlabeled* vertices.

Fix an $n \geq 1$, and consider a partition μ of length n as an n -dimensional vector

$$\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_+^n$$

consisting of positive integers. The Laplace transform of $H_g(\mu)$ as a function in μ ,

$$(6.8) \quad H_{g,n}(w_1, \dots, w_n) = \sum_{\mu \in \mathbb{Z}_+^n} H_g(\mu) e^{-(\mu_1(w_1+1) + \dots + \mu_n(w_n+1))},$$

is the function we wish to compute. Note that the automorphism group $\text{Aut}(\mu)$ acts trivially on the function $e^{-(\mu_1(w_1+1) + \dots + \mu_n(w_n+1))}$, which explains its appearance in (6.1). The reason for shifting the variables $w_i \mapsto w_i + 1$ is due to the asymptotic behavior

$$\frac{\mu^{\mu+k}}{\mu!} e^{-\mu} \sim \frac{1}{\sqrt{2\pi}} \mu^{k-\frac{1}{2}}$$

as μ approaches to ∞ . This asymptotics also suggests that the holomorphic function $H_{g,n}(w_1, \dots, w_n)$ is actually defined on a double-sheeted covering on the w_i -plane, since $\sqrt{w_i}$ behaves better as a holomorphic coordinate.

Following [24, 57], we introduce a series of polynomials $\hat{\xi}_n(t)$ of degree $2n + 1$ in t for $n \geq 0$ by the recursion formula

$$(6.9) \quad \hat{\xi}_n(t) = t^2(t-1) \frac{d}{dt} \hat{\xi}_{n-1}(t)$$

with the initial condition $\hat{\xi}_0(t) = t - 1$. This differential operator appears in [31]. The functions $\hat{\xi}_{-1}(t)$ and $\hat{\xi}_0(t)$ also appear as the two fundamental functions in [84] that generate his algebra \mathcal{A} . These polynomials are introduced to make the computation of the Laplace transform (6.8) easier.

Proposition 6.1 ([12, 24]). *Let us introduce new coordinates*

$$(6.10) \quad x = e^{-w}, \quad z = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu-1}}{\mu!} e^{-\mu} x^{\mu}, \quad t-1 = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu}}{\mu!} e^{-\mu} x^{\mu}.$$

Then the inverse function of $z = z(x)$ is given by

$$(6.11) \quad x = ze^{1-z},$$

and the variables z and t are related by

$$(6.12) \quad z = \frac{t-1}{t}.$$

Moreover, we have

$$(6.13) \quad \hat{\xi}_n(t) = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu+n}}{\mu!} e^{-\mu(w+1)} = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu+n}}{\mu!} e^{-\mu} x^{\mu}$$

for $n \geq 0$.

Proof. The infinite series (6.13) has the radius of convergence 1, and for $|x| < 1$, we can apply the Lagrange inversion formula to obtain (6.11). Since the application of

$$-\frac{d}{dw} = x \frac{d}{dx} = t^2(t-1) \frac{d}{dt}$$

n -times to $\sum_{\mu=1}^{\infty} \frac{\mu^{\mu}}{\mu!} e^{-\mu(w+1)}$ produces $\sum_{\mu=1}^{\infty} \frac{\mu^{\mu+n}}{\mu!} e^{-\mu(w+1)}$, we obtain (6.9). If we extend (6.13) formally to $n = -1$, then we have $z = \hat{\xi}_{-1}(t)$. To obtain the expression of z as a function of t , we need to solve the differential equation

$$t^2(t-1) \frac{d}{dt} \cdot z = t-1.$$

Its solution is $z = c - \frac{1}{t}$. Since $x = 0 \iff z = 0$ and $x = 0 \implies t = 1$, we conclude that the constant of integration is $c = 1$. Thus $z = 1 - 1/t$. \square

Remark 6.2. The relation between our z as a function in x and the classical Lambert W-function (see for example, [12]) is

$$z(x) = -W(-x/e).$$

Because of the ELSV formula (6.1), the Laplace transform of $H_g(\mu)$ becomes a polynomial in t_1, \dots, t_n for (g, n) in the stable range. The result is

$$(6.14) \quad \begin{aligned} F_{g,n}^H(t_1, \dots, t_n) &= H_{g,n}(w(t_1), \dots, w(t_n)) \\ &= \sum_{\mu \in \mathbb{Z}_+^n} H_g(\mu) e^{-(\mu_1(w_1+1) + \dots + \mu_n(w_n+1))} \\ &= \sum_{\mu \in \mathbb{Z}_+^n} \sum_{k_1 + \dots + k_n \leq 3g-3+n} \langle \tau_{k_1} \cdots \tau_{k_n} \Lambda_g^{\vee}(1) \rangle \prod_{i=1}^n \frac{\mu_i^{\mu_i+k_i}}{\mu_i!} e^{-(\mu_1(w_1+1) + \dots + \mu_n(w_n+1))} \\ &= \sum_{k_1 + \dots + k_n \leq 3g-3+n} \langle \tau_{k_1} \cdots \tau_{k_n} \Lambda_g^{\vee}(1) \rangle \prod_{i=1}^n \hat{\xi}_{k_i}(t_i). \end{aligned}$$

The Laplace transform (6.14) is no longer a polynomial for the unstable geometries $(g, n) = (0, 1)$ and $(0, 2)$. We use (6.5) to calculate $F_{0,1}^H$ and $F_{0,2}^H$.

Theorem 6.3. *The Laplace transform of the unstable cases $(g, n) = (0, 1)$ and $(0, 2)$ are given by*

$$(6.15) \quad F_{0,1}^H(t) = \frac{1}{2} \left(1 - \frac{1}{t^2} \right)$$

and

$$(6.16) \quad F_{0,2}^H(t_1, t_2) = \log \left(\frac{z_1 - z_2}{x_1 - x_2} \right) - (z_1 + z_2) + 1,$$

where t_i, x_i, z_i are related by (6.11) and (6.12).

Proof. The $(0, 1)$ case is a straightforward computation.

$$F_{0,1}^H(t) = \sum_{k=d}^{\infty} H_0((d)) e^{-d} x^d = \sum_{d=1}^{\infty} \frac{d^{d-2}}{d!} e^{-d} x^d = \hat{\xi}_{-2}(t).$$

This is a solution to the differential equation

$$t^2(t-1) \frac{d}{dt} \hat{\xi}_{-2}(t) = \hat{\xi}_{-1}(t) = z = \frac{t-1}{t}.$$

Therefore, $\hat{\xi}_{-2}(t) = c - \frac{1}{2} \frac{1}{t^2}$ for a constant of integration c . Here again we note

$$t = 1 \implies z = 0 \implies x = 0 \implies \hat{\xi}_{-2}(t) = 0.$$

This determines that $c = \frac{1}{2}$. Thus we have established (6.15).

Since

$$F_{0,2}^H(t_1, t_2) = \sum_{\mu_1, \mu_2 \geq 1} \frac{1}{\mu_1 + \mu_2} \cdot \frac{\mu_1^{\mu_1}}{\mu_1!} e^{-\mu_1} \cdot \frac{\mu_2^{\mu_2}}{\mu_2!} e^{-\mu_2} \cdot x_1^{\mu_1} x_2^{\mu_2}$$

and since $z = \hat{\xi}_{-1}(t)$, (6.16) is equivalent to

$$(6.17) \quad \sum_{\substack{\mu_1, \mu_2 \geq 0 \\ (\mu_1, \mu_2) \neq (0,0)}} \frac{1}{\mu_1 + \mu_2} \cdot \frac{\mu_1^{\mu_1}}{\mu_1!} e^{-\mu_1} \cdot \frac{\mu_2^{\mu_2}}{\mu_2!} e^{-\mu_2} \cdot x_1^{\mu_1} x_2^{\mu_2} = \log \left(e \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} e^{-k} \cdot \frac{x_1^k - x_2^k}{x_1 - x_2} \right),$$

where $|x_1| < 1, |x_2| < 1$, and $0 < |x_1 - x_2| < 1$ so that the formula is an equation of holomorphic functions in x_1 and x_2 . Define

$$\phi(x_1, x_2)$$

$$\stackrel{\text{def}}{=} \sum_{\substack{\mu_1, \mu_2 \geq 0 \\ (\mu_1, \mu_2) \neq (0,0)}} \frac{1}{\mu_1 + \mu_2} \cdot \frac{\mu_1^{\mu_1}}{\mu_1!} e^{-\mu_1} \cdot \frac{\mu_2^{\mu_2}}{\mu_2!} e^{-\mu_2} \cdot x_1^{\mu_1} x_2^{\mu_2} - \log \left(\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} e^{1-k} \cdot \frac{x_1^k - x_2^k}{x_1 - x_2} \right).$$

Then

$$\begin{aligned} \phi(x, 0) &= \sum_{\mu_1 \geq 1} \frac{\mu_1^{\mu_1-1}}{\mu_1!} e^{-\mu_1} x^{\mu_1} - \log \left(\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} e^{-k} \cdot x^{k-1} \right) - 1 \\ &= \hat{\xi}_{-1}(t) - \log \left(\frac{\hat{\xi}_{-1}(t)}{x} \right) - 1 = 1 - \frac{1}{t} - \log \left(1 - \frac{1}{t} \right) + \log x - 1 \\ &= -\frac{1}{t} - \log \left(1 - \frac{1}{t} \right) - w = 0 \end{aligned}$$

because

$$x = e^{-w} = ze^{1-z} = \left(1 - \frac{1}{t}\right) e^{\frac{1}{t}}.$$

Here t is restricted on the domain $Re(t) > 1$. Since

$$\begin{aligned} x_1 \frac{\partial}{\partial x_1} \log \left(e \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} e^{-k} \cdot \frac{x_1^k - x_2^k}{x_1 - x_2} \right) \\ = t_1^2 (t_1 - 1) \frac{\partial}{\partial t_1} \log \left(\hat{\xi}_{-1}(t_1) - \hat{\xi}_{-1}(t_2) \right) - x_1 \frac{\partial}{\partial x_1} \log(x_1 - x_2) \\ = t_1^2 (t_1 - 1) \frac{\partial}{\partial t_1} \log \left(-\frac{1}{t_1} + \frac{1}{t_2} \right) - \frac{x_1}{x_1 - x_2} \\ = \frac{t_1 t_2 (t_1 - 1)}{t_1 - t_2} - \frac{x_1}{x_1 - x_2}, \end{aligned}$$

we have

$$\begin{aligned} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \log \left(e \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} e^{-k} \cdot \frac{x_1^k - x_2^k}{x_1 - x_2} \right) \\ = \frac{t_1 t_2 (t_1 - 1) - t_1 t_2 (t_2 - 1)}{t_1 - t_2} - \frac{x_1 - x_2}{x_1 - x_2} \\ = t_1 t_2 - 1 = \hat{\xi}_0(t_1) \hat{\xi}_0(t_2) + \hat{\xi}_0(t_1) + \hat{\xi}_0(t_2). \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \sum_{\substack{\mu_1, \mu_2 \geq 0 \\ (\mu_1, \mu_2) \neq (0,0)}} \frac{1}{\mu_1 + \mu_2} \cdot \frac{\mu_1^{\mu_1}}{\mu_1!} e^{-\mu_1} \cdot \frac{\mu_2^{\mu_2}}{\mu_2!} e^{-\mu_2} \cdot x_1^{\mu_1} x_2^{\mu_2} \\ = \sum_{\substack{\mu_1, \mu_2 \geq 0 \\ (\mu_1, \mu_2) \neq (0,0)}} \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2} \cdot \frac{\mu_1^{\mu_1}}{\mu_1!} e^{-\mu_1} \cdot \frac{\mu_2^{\mu_2}}{\mu_2!} e^{-\mu_2} \cdot x_1^{\mu_1} x_2^{\mu_2} \\ = \hat{\xi}_0(t_1) \hat{\xi}_0(t_2) + \hat{\xi}_0(t_1) + \hat{\xi}_0(t_2). \end{aligned}$$

Therefore,

$$(6.18) \quad \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \phi(x_1, x_2) = 0.$$

Note that $\phi(x_1, x_2)$ is a holomorphic function in x_1 and x_2 . Therefore, it has a series expansion in homogeneous polynomials around $(0, 0)$. Since a homogeneous polynomial in x_1 and x_2 of degree n is an eigenvector of the differential operator $x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$ belonging to the eigenvalue n , the only holomorphic solution to the Euler differential equation (6.18) is a constant. But since $\phi(x_1, 0) = 0$, we conclude that $\phi(x_1, x_2) = 0$. This completes the proof of (6.17), and hence the proposition. \square

Definition 6.4. We define the symmetric differential forms for all $g \geq 0$ and $n > 0$ by

$$(6.19) \quad W_{g,n}^H(t_1, \dots, t_n) = d_1 \cdots d_n F_{g,n}^H(t_1, \dots, t_n),$$

and call them the Hurwitz differential forms.

The unstable cases are given by

$$(6.20) \quad W_{0,1}^H(t_1) = d_1 F_{0,1}^H(t_1) = \frac{1}{t_1^3} dt_1 = \frac{z}{x} dx,$$

and

$$(6.21) \quad \begin{aligned} W_{0,2}^H(t_1, t_2) &= d_1 d_2 F_{0,2}^H(t_1, t_2) = d_1 d_2 [\log(z_1 - z_2) - \log(x_1 - x_2)] \\ &= d_1 d_2 \left[\log\left(\frac{1}{t_2} - \frac{1}{t_1}\right) - \log(x_1 - x_2) \right] = d_1 d_2 [\log(t_1 - t_2) - \log(x_1 - x_2)] \\ &= \frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2} - \frac{dx_1 \cdot dx_2}{(x_1 - x_2)^2}. \end{aligned}$$

We note that all quantities are expressible in terms of z , or equivalently, in t . Now Definition 2.1 tells us that the spectral curve Σ of the single Hurwitz number is

$$(6.22) \quad \begin{cases} x = ze^{1-z} \\ y = \frac{z}{x} = e^{z-1}. \end{cases}$$

The *Lambert curve* Σ defined by $x = ze^{1-z}$, which is obtained by the Laplace transform of the number of trees, is an analytic curve and its x -projection has a simple ramification point at $z = 1$, since

$$dx = (1 - z)e^{1-z} dz.$$

The t -coordinate brings this ramification point to $t = \infty$. Let \bar{z} (resp. \bar{t}) denote the unique local Galois conjugate of z (reps. t). We also use

$$(6.23) \quad \bar{t} = s(t),$$

which is defined by the functional equation

$$(6.24) \quad \left(1 - \frac{1}{t}\right) e^{\frac{1}{t}} = \left(1 - \frac{1}{s(t)}\right) e^{\frac{1}{s(t)}}.$$

Although the Galois conjugate is only locally defined near the branched point $t = \infty$, we consider $s(t)$ as a global holomorphic function via analytic continuation. For $Re(t) > 1$, (6.24) implies

$$w(t) = -\log x = -\left(\frac{1}{t} - \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{t^n}\right) = \sum_{n=2}^{\infty} \frac{1}{t^n}.$$

When considered as a functional equation, (6.24) has exactly two solutions: t and

$$(6.25) \quad s(t) = -t + \frac{2}{3} + \frac{4}{135}t^{-2} + \frac{8}{405}t^{-3} + \frac{8}{567}t^{-4} + \dots.$$

This is the deck-transformation of the projection $\pi : \Sigma \rightarrow \mathbb{C}$ near $t = \infty$ and satisfies the involution equation $s(s(t)) = t$. It is analytic on $\mathbb{C} \setminus [0, 1]$ and has logarithmic singularities at 0 and 1.

Let us calculate the recursion kernel. Since

$$\frac{dx}{x} = \frac{1-z}{z} dz = \frac{dt}{t^2(t-1)} = \frac{s'(t)dt}{s(t)^2(s(t)-1)},$$

we have

$$(6.26) \quad K^H(t, t_1) = \frac{1}{2} \frac{\int_t^{s(t)} W_{0,2}^H(\cdot, t_1)}{W_{0,1}(s(t)) - W_{0,1}(t)} = \frac{1}{2} \left(\frac{1}{t - t_1} - \frac{1}{s(t) - t_1} \right) \frac{t^2(t-1)}{\frac{1}{t} - \frac{1}{s(t)}} \cdot \frac{1}{dt} \cdot dt_1$$

$$= \frac{1}{2} \left(\frac{1}{t-t_1} - \frac{1}{s(t)-t_1} \right) \frac{ts(t)}{s(t)-t} \cdot \frac{t^2(t-1)}{dt} \cdot dt_1.$$

Theorem 6.5 ([24, 57]). *The Hurwitz differential forms (6.19) for $2g-2+n > 0$ satisfy the Eynard-Orantin recursion:*

$$(6.27) \quad W_{g,n}^H(t_1, \dots, t_n) = \frac{1}{2\pi i} \oint_{\gamma_\infty} K^H(t, t_1) \left[W_{g-1, n+1}^H(t, s(t), t_2, \dots, t_n) \right. \\ \left. + \sum_{\substack{\text{No } (0,1)\text{-terms} \\ g_1+g_2=g \\ I \sqcup J = \{2, \dots, n\}}} W_{g_1, |I|+1}^H(t, t_I) W_{g_2, |J|+1}^H(s(t), t_J) \right],$$

where γ_∞ is a negatively oriented circle around ∞ whose radius is larger than any of $|t_j|$'s and $|s(t_j)|$'s.

Remark 6.6. The recursion formula (6.27) was first conjectured by Bouchard and Mariño in [7]. Its proofs appear in [4, 24, 57]. The method of [4] is to use a matrix integral expression of the single Hurwitz numbers. The idea of [24, 57] is that the Laplace transform of the *cut-and-join equation* of [29, 77] is the Eynard-Orantin recursion. The cut-and-join equation takes the following form:

$$(6.28) \quad r(g, \mu) H_g(\mu) = \sum_{i < j} (\mu_i + \mu_j) H_g(\mu(\hat{i}, \hat{j}), \mu_i + \mu_j) \\ + \frac{1}{2} \sum_{i=1}^n \sum_{\alpha+\beta=\mu_i} \alpha\beta \left(H_{g-1}(\mu(\hat{i}), \alpha, \beta) + \sum_{\substack{g_1+g_2=g \\ \nu_1 \sqcup \nu_2 = \mu(\hat{i})}} H_{g_1}(\nu_1, \alpha) H_{g_2}(\nu_2, \beta) \right).$$

Here μ is a partition of length n , and $\mu(\hat{i})$ and $\mu(\hat{i}, \hat{j})$ indicate the partition obtained by deleting parts of μ .

Remark 6.7. As we have seen above, Hurwitz numbers in (6.5) determine the shape of the recursion formula (6.27). Since the recursion gives the Hurwitz numbers for all (g, n) , we have thus established that unstable $(g, n) = (0, 1)$ and $(0, 2)$ Hurwitz numbers determine all other single Hurwitz numbers.

It is important to check if the formulas (2.4) and (2.6) agree with the geometry. From the definition (6.14) we calculate

$$F_{0,3}^H(t_1, t_2, t_3) = \langle \tau_0 \tau_0 \tau_0 \rangle_{0,3} \hat{\xi}_0(t_1) \hat{\xi}_0(t_2) \hat{\xi}_0(t_3) = (t_1-1)(t_2-1)(t_3-1),$$

which yields

$$(6.29) \quad W_{0,3}^H(t_1, t_2, t_3) = dt_1 dt_2 dt_3.$$

Since

$$dx(z) \cdot dy(z) = (1-z)dz \cdot dz = \frac{dt \cdot dt}{t^5}$$

from (6.22) and (6.11), the general formula (2.6) yields

$$W_{0,3}^H(t_1, t_2, t_3) = -\frac{1}{2\pi i} \oint_{\gamma_\infty} \frac{W_{0,2}^H(t, t_1) W_{0,2}^H(t, t_2) W_{0,2}^H(t, t_3)}{dx(t) \cdot dy(t)}$$

$$= - \left[\frac{1}{2\pi i} \oint_{\gamma_\infty} \frac{t^5}{(t-t_1)^2(t-t_2)^2(t-t_3)^2} dt \right] dt_1 dt_2 dt_3 = dt_1 dt_2 dt_3,$$

in agreement with geometry. Here we calculate the residue at $t = \infty$. Although

$$W_{0,2}^H(t, t_i) = \frac{dt \cdot dt_i}{(t-t_i)^2} - \frac{dx \cdot dx_i}{(x-x_i)^2},$$

the second term does not contribute to the integral. This is because as $t \rightarrow \infty$, we have $x \rightarrow 1$, and $dx \cdot dx_i / (x-x_i)^2$ has no pole at $x = 1$.

Similarly,

$$F_{1,1}^H(t_1) = \langle \tau_1 \rangle_{1,1} \hat{\xi}_1(t_1) - \langle \tau_0 \lambda_1 \rangle_{1,1} \hat{\xi}_0(t_1) = \frac{1}{24} (t_1^2 - 1)(t_1 - 1),$$

and thus we have

$$(6.30) \quad W_{1,1}^H(t_1) = \frac{1}{24} (t_1 - 1)(3t_1 + 1) dt_1.$$

On the other hand, the general formula (2.4) gives

$$\begin{aligned} W_{1,1}^H(t_1) &= \frac{1}{2\pi i} \oint_{\gamma_\infty} K^H(t, t_1) \left[W_{0,2}^H(u, v) + \frac{dx(u) \cdot dx(v)}{(x(u) - x(v))^2} \right] \Big|_{\substack{u=t \\ v=s(t)}} \\ &= \frac{1}{2\pi i} \oint_{\gamma_\infty} K^H(t, t_1) \frac{dt \cdot s'(t) dt}{(t - s(t))^2} \\ &= \left[\frac{1}{2\pi i} \oint_{\gamma_\infty} \frac{1}{2} \left(\frac{1}{t-t_1} - \frac{1}{s(t)-t_1} \right) \frac{ts(t)}{s(t)-t} t^2 (t-1) \frac{s'(t) dt}{(t-s(t))^2} \right] dt_1 \\ &= \frac{t_1 s(t_1)}{(t_1 - s(t_1))^3} s(t_1)^2 (s(t_1) - 1) dt_1 \\ &\quad - \left[\frac{1}{2\pi i} \oint_{\gamma_{[0,1]}} \frac{1}{2} \left(\frac{1}{t-t_1} - \frac{1}{s(t)-t_1} \right) \frac{ts(t)}{s(t)-t} t^2 (t-1) \frac{s'(t) dt}{(t-s(t))^2} \right] dt_1, \end{aligned}$$

where $\gamma_{[0,1]}$ is a contour circling around the slit $[0, 1]$ in the t -plane in the positive direction.

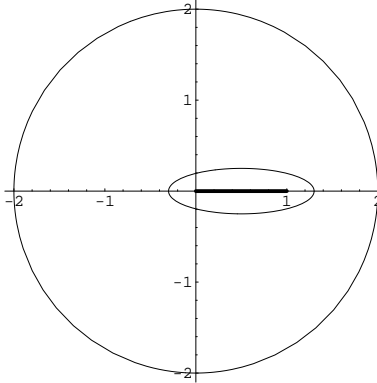


FIGURE 6.2. The contours of integration. The outer loop γ_∞ is the circle of a large radius oriented clock wise, and $\gamma_{[0,1]}$ is the thin loop surrounding the closed interval $[0, 1]$ in the positive direction.

Note that the integrand of the last integral is a holomorphic function in t on $\gamma_{[0,1]}$, hence it has a finite value. It is also clear that as $t_1 \rightarrow \infty$, this integral tends to 0, because $\gamma_{[0,1]}$ is a compact space. Therefore, we conclude that

$$\begin{aligned} W_{1,1}^H(t_1) &= \frac{t_1 s(t_1)}{(t_1 - s(t_1))^3} s(t_1)^2 (s(t_1) - 1) dt_1 + O(1/t_1) \\ &= \left(\frac{1}{8} t_1^2 - \frac{1}{12} t_1 - \frac{1}{24} \right) dt_1 + O(1/t_1), \end{aligned}$$

since $s(t) = -t + 2/3 + O(1/t^2)$. It agrees with (6.30) because of the following

Lemma 6.8. *A solution to the topological recursion (6.27) is a polynomial in t_1 .*

Proof. The t_1 -dependence of $W_{g,n}^H(t_1, \dots, t_n)$ only comes from the factor

$$\begin{aligned} \left(\frac{1}{t - t_1} - \frac{1}{s(t) - t_1} \right) \\ = \frac{1}{t} + \frac{1}{3} \frac{1}{t^2} + \left(t_1^2 - \frac{2}{3} t_1 + \frac{2}{9} \right) \frac{1}{t^3} + \left(t_1^2 - \frac{2}{3} t_1 + \frac{22}{135} \right) \frac{1}{t^4} + \dots \end{aligned}$$

in the recursion kernel (6.26). Since each coefficient of the t -expansion of $K^H(t, t_1)$ is a polynomial in t_1 , Lemma follows. \square

7. THE STATIONARY GROMOV-WITTEN INVARIANTS OF \mathbb{P}^1

In this section we study the generating functions of stationary Gromov-Witten invariants of \mathbb{P}^1 . The conjectural relation between these invariants and the Eynard-Orantin topological recursion was first formulated in [62]. We identify the spectral curve and the recursion kernel using the unstable geometries.

Morally speaking, the space \mathbb{P}^1 we are considering here appears as the zero section of a Calabi-Yau threefold known as the *resolved conifold* X , which is the total space of the rank 2 vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ over \mathbb{P}^1 . Let $L \subset X$ be a special Lagrangian submanifold [39, 42]. Then the intersection $L \cap \mathbb{P}^1$ of the special Lagrangian and the zero section is a circle on \mathbb{P}^1 . If we holomorphically embed a bordered Riemann surface with n boundary components into X in a way that each boundary is mapped to a distinct circle on \mathbb{P}^1 , then the whole Riemann surface is necessarily mapped to \mathbb{P}^1 . Thus we are considering *open* Gromov-Witten invariants of \mathbb{P}^1 . And if we make these circles on \mathbb{P}^1 small and centered around n distinct points of \mathbb{P}^1 , then we are naturally led to the stationary Gromov-Witten invariants of \mathbb{P}^1 .

So our main object of this section is the Laplace transform of the stationary Gromov-Witten invariants

$$(7.1) \quad F_{g,n}^{\mathbb{P}^1}(x_1, \dots, x_n) = \sum_{\mu_1, \dots, \mu_n=0}^{\infty} \langle \tau_{\mu_1}(\omega) \cdots \tau_{\mu_n}(\omega) \rangle_{g,n} \prod_{i=1}^n \mu_i! \prod_{i=1}^n \frac{1}{x^{\mu_i+1}},$$

where $\omega \in A_0(\mathbb{P}^1)$ is the point class generator, and

$$(7.2) \quad \langle \tau_{\mu_1}(\omega) \cdots \tau_{\mu_n}(\omega) \rangle_{g,n} = \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)]^{\text{virt}}} \psi_1^{\mu_1} ev_1^*(\omega) \cdots \psi_1^{\mu_n} ev_n^*(\omega)$$

is a stationary Gromov-Witten invariant of \mathbb{P}^1 . More precisely, $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ is the moduli stack of stable morphisms from a connected n -pointed curve (C, p_1, \dots, p_n) into \mathbb{P}^1 of degree

d such that $f(p_i)$, $i = 1, \dots, n$, are distinct, and ev_i is the natural evaluation morphism

$$ev_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \ni [f, (C, p_1, \dots, p_n)] \longmapsto f(p_i) \in \mathbb{P}^1.$$

The Gromov-Witten invariant (7.2) vanishes unless

$$(7.3) \quad 2g - 2 + 2d = \mu_1 + \dots + \mu_n.$$

The sum in (7.1) is the Laplace transform if we identify

$$(7.4) \quad x = e^w.$$

The extra numerical factor $\prod_{i=1}^n \mu_i!$ is included in (7.1) because of the polynomial growth order of

$$(7.5) \quad \langle \tau_{\mu_1}(\omega) \cdots \tau_{\mu_n}(\omega) \rangle_{g,n} \prod_{i=1}^n \mu_i!$$

for large μ that is established in [66]. Indeed (7.5) is essentially a special type of *Hurwitz numbers* that counts the number of certain coverings of \mathbb{P}^1 .

To determine the spectral curve and the annulus amplitude, we need to consider unstable geometries $(g, n) = (0, 1)$ and $(0, 2)$. From [66] we learn

$$(7.6) \quad \langle \tau_{\mu_1}(\omega) \rangle_{0,1} = \langle \tau_{2d-2}(\omega) \rangle_{0,1} = \left(\frac{1}{d!} \right)^2.$$

To compute a closed formula for

$$F_{0,1}^{\mathbb{P}^1}(x) = \sum_{\mu_1=0}^{\infty} \langle \tau_{\mu_1}(\omega) \rangle_{0,1} \mu_1! \frac{1}{x^{\mu_1+1}} = \sum_{d=1}^{\infty} \frac{(2d-2)!}{d!d!} \frac{1}{x^{2d-1}},$$

we notice that the generating function of Catalan numbers (3.5)

$$z(x) = \sum_{m=0}^{\infty} C_m \frac{1}{x^{2m+1}}$$

provides again an effective tool. Thus we have

$$(7.7) \quad \left(x \frac{d}{dx} - 1 \right) F_{0,1}^{\mathbb{P}^1}(x) = -2 \sum_{d=1}^{\infty} \frac{(2d-2)!}{(d-1)!d!} \frac{1}{x^{2d-1}} \\ = -2 \sum_{m=0}^{\infty} \frac{(2m)!}{(m+1)!m!} \frac{1}{x^{2m+1}} = -2z(x).$$

The advantage of using the Catalan series $z(x)$ is that we know its inverse function (3.8). Using (4.6), we see that (7.7) is equivalent to

$$(7.8) \quad \left(\frac{z^3 + z}{z^2 - 1} \frac{d}{dz} - 1 \right) F_{0,1}^{\mathbb{P}^1}(z) = -2z.$$

The solution of (7.8) is given by

$$F_{0,1}^{\mathbb{P}^1}(z) = -\frac{2}{z} - \left(z + \frac{1}{z} \right) \log(1 + z^2) + c \left(z + \frac{1}{z} \right),$$

with a constant of integration c . Since

$$z \rightarrow 0 \implies x \rightarrow \infty \implies F_{0,1}^{\mathbb{P}^1} \rightarrow 0,$$

we conclude that $c = 2$. We thus obtain

$$(7.9) \quad F_{0,1}^{\mathbb{P}^1}(z) = 2z - \left(z + \frac{1}{z}\right) \log(1 + z^2),$$

and therefore,

$$(7.10) \quad W_{0,1}^{\mathbb{P}^1}(z) = dF_{0,1}^{\mathbb{P}^1}(z) = -\log(1 + z^2) d\left(z + \frac{1}{z}\right).$$

Theorem 7.1. *The spectral curve for the stationary Gromov-Witten invariants of \mathbb{P}^1 is given by*

$$(7.11) \quad \begin{cases} x = z + \frac{1}{z} \\ y = -\log(1 + z^2). \end{cases}$$

Remark 7.2. Since $dx = 0$ has two zeros at $z = \pm 1$, we also use as our preferred coordinate

$$(7.12) \quad t = \frac{z+1}{z-1} \iff z = \frac{t+1}{t-1}.$$

The log singularity on the t -plane is the right semicircle of radius 1 connecting i to $-i$ (see Figure 7.1). The expression of $W_{0,1}^{\mathbb{P}^1}$ in terms of the preferred coordinate is

$$(7.13) \quad W_{0,1}^{\mathbb{P}^1}(t) = \frac{8t}{(t^2-1)^2} \log\left(\frac{2(t^2+1)}{(t-1)^2}\right) dt.$$

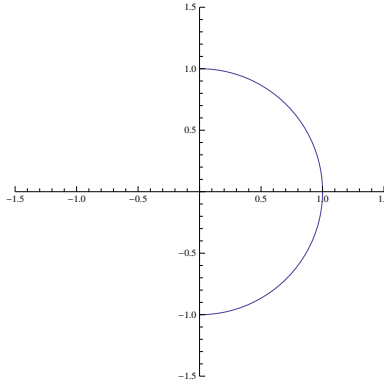


FIGURE 7.1. The spectral curve for the stationary Gromov-Witten invariants of \mathbb{P}^1 is the complex t -plane minus the semicircle.

Remark 7.3. The function $x = z + \frac{1}{z}$ is expected here, since it is the Landau-Ginzburg model that is homologically mirror dual to \mathbb{P}^1 [2].

Remark 7.4. The Galois conjugate of $x = z + \frac{1}{z}$ is globally defined, and is given by

$$(7.14) \quad t \longmapsto \bar{t} = -t.$$

Remark 7.5. Since

$$(7.15) \quad \frac{1}{1-z(x)} = \sum_{k=0}^{\infty} z(x)^k = 1 + \sum_{n=0}^{\infty} \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{1}{x^{n+1}},$$

we can express t in the branch near $t = -1$ as a function in x . The result is

$$(7.16) \quad t + 1 = \frac{z(x) + 1}{z(x) - 1} + 1 = 2 - \frac{2}{1 - z(x)} = - \sum_{n=0}^{\infty} 2 \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{1}{x^{n+1}},$$

which is also absolutely convergent for $|x| > 2$.

Remark 7.6. We are using the normalized Gromov-Witten invariants (7.5) to compute the Laplace transform (7.1). If we did not include the $\mu!$ factor in our computation of the spectral curve, then we would have encountered with the *modified Bessel function*

$$I_0(2x) = \sum_{m=1}^{\infty} \frac{1}{(m!)^2} x^{2m},$$

instead of $z(x)$, in computing (7.9). We note that $I_0(2x)$ appears in [18] in the exact same context of computing the Gromov-Witten invariants of \mathbb{P}^1 . We prefer the Catalan number series $z(x)$ over the modified Bessel function mainly because the inverse function of $z(x)$ takes a simple form $x = z + \frac{1}{z}$.

Motivated by the technique developed in [7, 24, 57] for single Hurwitz numbers, let us define

$$(7.17) \quad \xi_n(t) = \sum_{k=0}^{\infty} \binom{2k}{k} k^n \frac{1}{x^{2k+1}}, \quad n \geq 0,$$

and

$$(7.18) \quad \eta_n(t) = \sum_{k=0}^{\infty} \binom{2k+1}{k} k^n \frac{1}{x^{2k+2}}, \quad n \geq 0.$$

We then have

$$(7.19) \quad \xi_{n+1}(t) = -\frac{1}{2} \left(x \frac{d}{dx} + 1 \right) \xi_n(t) = \left(\frac{t^4 - 1}{8t} \frac{d}{dt} - \frac{1}{2} \right) \xi_n(t)$$

and

$$(7.20) \quad \eta_{n+1}(t) = -\frac{1}{2} \left(x \frac{d}{dx} + 2 \right) \eta_n(t) = \left(\frac{t^4 - 1}{8t} \frac{d}{dt} - 1 \right) \eta_n(t).$$

The initial values are computed as follows:

$$(7.21) \quad \begin{aligned} \xi_0(t) &= \frac{1}{2} \left(1 - x \frac{d}{dx} \right) \sum_{m=0}^{\infty} \frac{1}{m+1} \binom{2m}{m} \frac{1}{x^{2m+1}} \\ &= \frac{1}{2} \left(1 - \frac{z(z^2+1)}{z^2-1} \frac{d}{dz} \right) z = -\frac{z}{z^2-1} = -\frac{t^2-1}{4t}, \end{aligned}$$

and similarly

$$(7.22) \quad \eta_0(t) = -\frac{(t+1)^2}{4t}.$$

We note that $\xi_n(t)$ and $\eta_n(t)$ are Laurent polynomials of degree $2n+1$ for every $n \geq 0$. Since they are defined as functions in x , we have the reciprocity property

$$(7.23) \quad \begin{aligned} \xi_n(1/t) &= -\xi_n(t) \\ \eta_n(1/t) &= \eta_n(t). \end{aligned}$$

This follows from

$$t \mapsto \frac{1}{t} \implies x \mapsto -x.$$

The annulus amplitude requires $(g, n) = (0, 2)$ Gromov-Witten invariants. They can be calculated from the $(g, n) = (0, 1)$ invariants using the Topological Recursion Relation [28]. The results are

$$(7.24) \quad \langle \tau_{\mu_1}(\omega) \tau_{\mu_2}(\omega) \rangle_{0,2} = \begin{cases} \frac{1}{(m_1!)^2 (m_2!)^2} \frac{1}{(m_1+m_2+1)} & \mu_1 = 2m_1, \mu_2 = 2m_2 \\ \frac{1}{(m_1!)^2 (m_2!)^2} \frac{1}{(m_1+m_2+2)} & \mu_1 = 2m_1 + 1, \mu_2 = 2m_2 + 1. \end{cases}$$

Theorem 7.7. *The annulus amplitude is given by*

$$(7.25) \quad F_{0,2}^{\mathbb{P}^1}(z_1, z_2) = -\log(1 - z_1 z_2).$$

Hence we have

$$(7.26) \quad W_{0,2}^{\mathbb{P}^1}(t_1, t_2) = \frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2} - \frac{dx_1 \cdot dx_2}{(x_1 - x_2)^2} = \frac{dt_1 \cdot dt_2}{(t_1 + t_2)^2}.$$

Proof. From (7.24) we calculate

$$\begin{aligned} F_{0,2}^{\mathbb{P}^1}(z_1, z_2) &= \sum_{\mu_1, \mu_2=0}^{\infty} \langle \tau_{\mu_1}(\omega) \tau_{\mu_2}(\omega) \rangle_{0,2} \mu_1! \mu_2! \frac{1}{x_1^{\mu_1+1}} \frac{1}{x_2^{\mu_2+1}} \\ &= \sum_{m_1, m_2=0}^{\infty} \frac{1}{(m_1 + m_2 + 1)} \binom{2m_1}{m_1} \binom{2m_2}{m_2} \frac{1}{x_1^{2m_1+1}} \frac{1}{x_2^{2m_2+1}} \\ &\quad + \sum_{m_1, m_2=0}^{\infty} \frac{1}{(m_1 + m_2 + 2)} (2m_1 + 1)(2m_2 + 1) \binom{2m_1}{m_1} \binom{2m_2}{m_2} \frac{1}{x_1^{2m_1+2}} \frac{1}{x_2^{2m_2+2}}. \end{aligned}$$

Thus we have

$$\begin{aligned} &\left(x_1 \frac{d}{dx_1} + x_2 \frac{d}{dx_2} \right) F_{0,2}^{\mathbb{P}^1}(z_1, z_2) \\ &= -2 \sum_{m_1, m_2=0}^{\infty} \binom{2m_1}{m_1} \binom{2m_2}{m_2} \frac{1}{x_1^{2m_1+1}} \frac{1}{x_2^{2m_2+1}} \\ &\quad - 2 \sum_{m_1, m_2=0}^{\infty} (2m_1 + 1)(2m_2 + 1) \binom{2m_1}{m_1} \binom{2m_2}{m_2} \frac{1}{x_1^{2m_1+2}} \frac{1}{x_2^{2m_2+2}} \\ &= -2\xi_0(x_1)\xi_0(x_2) - 2z'(x_1)z'(x_2) \\ &= -2 \frac{z_1}{z_1^2 - 1} \frac{z_2}{z_2^2 - 1} - 2 \frac{z_1^2}{z_1^2 - 1} \frac{z_2^2}{z_2^2 - 1} = -2 \frac{z_1 z_2 (1 + z_1 z_2)}{(z_1^2 - 1)(z_2^2 - 1)}, \end{aligned}$$

where $\xi_0(x)$ is calculated in (7.21), and from (4.6) we know

$$z'(x) = \frac{dz}{dx} = \frac{z^2}{z^2 - 1}.$$

On the other hand,

$$\begin{aligned} &\left(x_1 \frac{d}{dx_1} + x_2 \frac{d}{dx_2} \right) (-\log(1 - z_1 z_2)) \\ &= \left(\frac{z_1(z_1^2 + 1)}{z_1^2 - 1} \frac{d}{dz_1} + \frac{z_2(z_2^2 + 1)}{z_2^2 - 1} \frac{d}{dz_2} \right) (-\log(1 - z_1 z_2)) \end{aligned}$$

$$= \left(\frac{(z_1^2 + 1)}{z_1^2 - 1} + \frac{(z_2^2 + 1)}{z_2^2 - 1} \right) \frac{z_1 z_2}{1 - z_1 z_2} = -2 \frac{z_1 z_2 (1 + z_1 z_2)}{(z_1^2 - 1)(z_2^2 - 1)}.$$

Therefore,

$$\begin{aligned} & \left(x_1 \frac{d}{dx_1} + x_2 \frac{d}{dx_2} \right) \left(F_{0,2}^{\mathbb{P}^1}(z_1, z_2) + \log(1 - z_1 z_2) \right) \\ &= \left(x_1 \frac{d}{dx_1} + x_2 \frac{d}{dx_2} \right) \left(\sum_{\mu_1, \mu_2=0}^{\infty} \langle \tau_{\mu_1}(\omega) \tau_{\mu_2}(\omega) \rangle_{0,2} \mu_1! \mu_2! \frac{1}{x_1^{\mu_1+1}} \frac{1}{x_2^{\mu_2+1}} \right. \\ & \quad \left. - \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=0}^{\infty} C_m \frac{1}{x_1^{2m+1}} \sum_{m=0}^{\infty} C_m \frac{1}{x_2^{2m+1}} \right)^n \right) = 0. \end{aligned}$$

Since the kernel of the Euler differential operator is the constants, and since actual computation shows that the first few expansion terms of the Laurent series

$$\sum_{\mu_1, \mu_2=0}^{\infty} \langle \tau_{\mu_1}(\omega) \tau_{\mu_2}(\omega) \rangle_{0,2} \mu_1! \mu_2! \frac{1}{x_1^{\mu_1+1}} \frac{1}{x_2^{\mu_2+1}} - \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=0}^{\infty} C_m \frac{1}{x_1^{2m+1}} \sum_{m=0}^{\infty} C_m \frac{1}{x_2^{2m+1}} \right)^n$$

are 0, we complete the proof of (7.25). \square

Using $\xi_n(t)$ and $\eta_n(t)$ of (7.17) and (7.18) and the classical topological recursion relation [28], we can systematically calculate the Laplace transform of stationary Gromov-Witten invariants. First let us consider $(g, n) = (0, 3)$. Since the sum of the descendant indices of

$$\langle \tau_{\mu_1}(\omega) \tau_{\mu_2}(\omega) \tau_{\mu_3}(\omega) \rangle_{0,3}$$

is even, we have

$$(7.27) \quad \begin{aligned} \langle \tau_{2m_1}(\omega) \tau_{2m_2}(\omega) \tau_{2m_3}(\omega) \rangle_{0,3} &= \frac{1}{m_1^2 m_2^2 m_3^2}, \\ \langle \tau_{2m_1}(\omega) \tau_{2m_2+1}(\omega) \tau_{2m_3+1}(\omega) \rangle_{0,3} &= \frac{(m_2 + 1)(m_3 + 1)}{m_1^2 (m_2 + 1)^2 (m_3 + 1)^2}. \end{aligned}$$

The Laplace transform is therefore

$$(7.28) \quad \begin{aligned} F_{0,3}^{\mathbb{P}^1}(t_1, t_2, t_3) &= \sum_{\mu_1 \mu_2 \mu_3 \geq 0} \langle \tau_{\mu_1}(\omega) \tau_{\mu_2}(\omega) \tau_{\mu_3}(\omega) \rangle_{0,3} \mu_1! \mu_2! \mu_3! \frac{1}{x_1^{\mu_1+1}} \cdot \frac{1}{x_2^{\mu_2+1}} \cdot \frac{1}{x_3^{\mu_3+1}} \\ &= \sum_{m_1, m_2, m_3 \geq 0} \binom{2m_1}{m_1} \binom{2m_2}{m_2} \binom{2m_3}{m_3} \frac{1}{x_1^{2m_1+1}} \cdot \frac{1}{x_2^{2m_2+1}} \cdot \frac{1}{x_3^{2m_3+1}} \\ &+ \sum_{m_1, m_2, m_3 \geq 0} \binom{2m_1}{m_1} \binom{2m_2+1}{m_2} \binom{2m_3+1}{m_3} \frac{1}{x_1^{2m_1+1}} \cdot \frac{1}{x_2^{2m_2+2}} \cdot \frac{1}{x_3^{2m_3+2}} \\ &+ \sum_{m_1, m_2, m_3 \geq 0} \binom{2m_1+1}{m_1} \binom{2m_2}{m_2} \binom{2m_3+1}{m_3} \frac{1}{x_1^{2m_1+2}} \cdot \frac{1}{x_2^{2m_2+1}} \cdot \frac{1}{x_3^{2m_3+2}} \\ &+ \sum_{m_1, m_2, m_3 \geq 0} \binom{2m_1+1}{m_1} \binom{2m_2+1}{m_2} \binom{2m_3}{m_3} \frac{1}{x_1^{2m_1+2}} \cdot \frac{1}{x_2^{2m_2+2}} \cdot \frac{1}{x_3^{2m_3+1}} \\ &= \xi_0(t_1) \xi_0(t_2) \xi_0(t_3) + \xi_0(t_1) \eta_0(t_2) \eta_0(t_3) + \eta_0(t_1) \xi_0(t_2) \eta_0(t_3) + \eta_0(t_1) \eta_0(t_2) \xi_0(t_3) \\ &= -\frac{1}{16} (t_1 + 1)(t_2 + 1)(t_3 + 1) \left(1 - \frac{1}{t_1 t_2 t_3} \right), \end{aligned}$$

which is indeed a Laurent polynomial. Since it is an odd degree polynomial in $\xi_n(t)$'s, we have the reciprocity

$$F_{0,3}^{\mathbb{P}^1}(1/t_1, 1/t_2, 1/t_3) = -F_{0,3}^{\mathbb{P}^1}(t_1, t_2, t_3).$$

The $n = 1$ stationary invariants are concretely calculated in [66]. We have

$$(7.29) \quad \begin{aligned} \langle \tau_{2d} \rangle_{1,1} &= \frac{1}{24} \left(\frac{1}{d!} \right)^2 (2d-1) \\ \langle \tau_{2d+2} \rangle_{2,1} &= \left(\frac{1}{d!} \right)^2 \left(\frac{1}{5! 4^2} (2d-1) + \frac{1}{24^2} \binom{2d-1}{2} \right) \\ \langle \tau_{2g-2+2d} \rangle_{g,1} &= \left(\frac{1}{d!} \right)^2 \sum_{\ell=1}^g \binom{2d-1}{\ell} \sum_{\substack{k_i > 0 \\ k_1 + \dots + k_\ell = g}} \prod_{i=1}^{\ell} \frac{1}{(2k_i + 1)! 4^{k_i}}. \end{aligned}$$

We thus obtain

$$(7.30) \quad \begin{aligned} F_{1,1}^{\mathbb{P}^1}(t_1) &= \frac{1}{24} \sum_{d=0}^{\infty} \binom{2d}{d} (2d-1) \frac{1}{x_1^{2d+1}} = \frac{1}{24} (2\xi_1(t_1) - \xi_0(t_1)) \\ &= -\frac{1}{384} \left(t_1^3 - 7t_1 + \frac{7}{t_1} - \frac{1}{t_1^3} \right). \end{aligned}$$

To calculate the $g = 2$ case we need to do the following.

$$(7.31) \quad \begin{aligned} F_{2,1}^{\mathbb{P}^1}(t_1) &= \sum_{d=0}^{\infty} \frac{(2d+2)!}{d! d!} \left(\frac{1}{5! 4^2} (2d-1) + \frac{1}{24^2} \binom{2d-1}{2} \right) \frac{1}{x_1^{2d+3}} \\ &= \left(\frac{d}{dx_1} \right)^2 \sum_{d=0}^{\infty} \binom{2d}{d} \left(\frac{1}{5! 4^2} (2d-1) + \frac{1}{24^2} (2d^2 - 3d + 1) \right) \frac{1}{x_1^{2d+1}} \\ &= \left(-\frac{(t^2-1)^2}{8t} \frac{d}{dt} \right)^2 \left[\frac{1}{5! 4^2} (2\xi_1(t_1) - \xi_0(t_1)) + \frac{1}{24^2} (2\xi_2(t_1) - 3\xi_1(t_1) + \xi_0(t_1)) \right] \\ &= -\frac{1}{2^{19} \cdot 3^2 \cdot 5} \frac{(t^2-1)^3}{t^9} (525t_1^{12} - 1470t_1^{10} + 1107t_1^8 + 527t_1^6 + 1107t_1^4 - 1470t_1^2 + 525). \end{aligned}$$

Proposition 7.8. $F_{g,1}^{\mathbb{P}^1}(t_1)$ is a Laurent polynomial of degree $6g - 3$ with the reciprocity

$$F_{g,1}^{\mathbb{P}^1}(1/t_1) = -F_{g,1}^{\mathbb{P}^1}(t_1).$$

Proof. First we calculate the binomial coefficient

$$\binom{2d-1}{\ell} = \frac{1}{\ell!} (2d-1)(2d-2) \cdots (2d-\ell) = \frac{1}{\ell!} \left(2^\ell d^\ell - \frac{\ell(\ell+1)}{2} d^{\ell-1} + \cdots + (-1)^\ell \ell! \right)$$

as a polynomial in d , and then replace each d^i with $\xi_i(t_1)$. The result is a linear combination of $\xi_0(t_1), \dots, \xi_\ell(t_1)$. Let $\Xi_\ell(t_1)$ denote the resulting Laurent polynomial of degree $2\ell + 1$. Then we have an expression

$$(7.32) \quad F_{g,1}^{\mathbb{P}^1}(t_1) = \left(\frac{d}{dx_1} \right)^{2g-2} \sum_{d=0}^{\infty} \binom{2d}{d} \sum_{\ell=1}^g \binom{2d-1}{\ell} \sum_{\substack{k_i > 0 \\ k_1 + \dots + k_\ell = g}} \prod_{i=1}^{\ell} \frac{1}{(2k_i + 1)! 4^{k_i}} \frac{1}{x_1^{2d+1}}$$

$$= \left(-\frac{(t^2-1)^2}{8t} \frac{d}{dt} \right)^{2g-2} \left[\sum_{\ell=1}^g \Xi_{\ell}(t_1) \sum_{\substack{k_i > 0 \\ k_1 + \dots + k_{\ell} = g}} \prod_{i=1}^{\ell} \frac{1}{(2k_i+1)! 4^{k_i}} \right],$$

which is a Laurent polynomial of degree $2(2g-2) + 2g + 1 = 6g - 3$. The reciprocity property follows from (7.23) and the x_1 expression of (7.32), where x_1 changes to $-x_1$. In particular, the even order differentiation in x_1 is not affected by this change. \square

The Eynard-Orantin recursion is for the differential forms $W_{g,n}^{\mathbb{P}^1}(t_1, \dots, t_n)$. We need the recursion kernel. From (7.13) and (7.26), we compute

$$\begin{aligned} (7.33) \quad K^{\mathbb{P}^1}(t, t_1) &= \frac{1}{2} \frac{\int_t^{-t} W_{0,2}^{\mathbb{P}^1}(\cdot, t_1)}{W_{0,1}^{\mathbb{P}^1}(-t) - W_{0,1}^{\mathbb{P}^1}(t)} \\ &= \frac{1}{2} \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{1}{\log\left(\frac{2(t^2+1)}{(t+1)^2}\right) - \log\left(\frac{2(t+1)}{(t-1)^2}\right)} \frac{(t^2-1)^2}{-8tdt} \cdot dt_1 \\ &= \frac{1}{16} \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{1}{\log\left(\frac{(t-1)^2}{(t+1)^2}\right)} \frac{(t^2-1)^2}{tdt} \cdot dt_1. \end{aligned}$$

We note the reciprocity property of the kernel

$$(7.34) \quad K^{\mathbb{P}^1}(1/t, 1/t_1) = -K^{\mathbb{P}^1}(t, t_1).$$

The topological recursion (2.3) becomes

$$\begin{aligned} (7.35) \quad W_{g,n}^{\mathbb{P}^1}(t_1, t_2, \dots, t_n) &= \frac{1}{2\pi i} \oint_{\gamma} K^{\mathbb{P}^1}(t, t_1) \left[W_{g-1, n+1}^{\mathbb{P}^1}(t, -t, t_2, \dots, t_n) \right. \\ &\quad \left. + \sum_{\substack{\text{No } (0,1) \text{ terms} \\ g_1+g_2=g \\ I \sqcup J = \{2,3,\dots,n\}}} W_{g_1, |I|+1}^{\mathbb{P}^1}(t, t_I) W_{g_2, |J|+1}^{\mathbb{P}^1}(-t, t_J) \right], \end{aligned}$$

where the residue calculation is taken along the integration contour γ (see Figure 4.1) consisting of two concentric circles of radius ϵ and $1/\epsilon$ for a small ϵ centered around $t = 0$, with the inner circle positively oriented and the outer circle negatively oriented. Since there is a log singularity in the complex t -plane, we cannot use the residue calculus method to evaluate the integral at $t = t_1$ and $t = -t_1$. Thus the residue calculation of (7.35) is performed around the neighborhood of $t = 0$ and $t = \infty$.

So let us provide two expansion formulas for the kernel $K^{\mathbb{P}^1}(t, t_1)$, assuming that $t_1 \in \mathbb{C}^*$ is away from the log singularity of Figure 7.1. The transcendental factor of $K^{\mathbb{P}^1}(t, t_1)$ has an expansion

$$(7.36) \quad \frac{4}{t \log\left(\frac{(t-1)^2}{(t+1)^2}\right)} = -\frac{1}{t^2} + \frac{1}{3} + \frac{4}{45}t^2 + \frac{44}{945}t^4 + \frac{428}{14175}t^6 + \frac{10196}{467775}t^8 + \dots$$

around $t = 0$. The denominator of the coefficient of t^{2k-2} is given by

$$\prod_{q=3, \text{ prime}}^{2k+1} q^{\lfloor \frac{2k}{q-1} \rfloor} = 3^{\lfloor k \rfloor} \cdot 5^{\lfloor \frac{k}{2} \rfloor} \cdot 7^{\lfloor \frac{k}{3} \rfloor} \dots,$$

which is the same as $\mu(L_k)$ of [38, Lemma 1.5.2]. The expansion of $\frac{1}{t+t_1} + \frac{1}{t-t_1}$ at $t = 0$ is given by

$$\frac{1}{t+t_1} + \frac{1}{t-t_1} = -2t \frac{1}{t_1^2} \frac{1}{1-\frac{t^2}{t_1^2}} = -2 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{t_1^{2n+2}}.$$

From the expression (7.33) and the above consideration, we know that around $t = 0$, $K^{\mathbb{P}^1}(t, t_1)$ starts from t^{-1} , and that the coefficient of t^{2n-1} is a Laurent polynomial in t_1^2 starting from $\frac{1}{32}t_1^{-(2n+2)}$ up to t_1^{-2} with rational coefficients. More concretely, we have

$$(7.37) \quad K^{\mathbb{P}^1}(t, t_1) = \left[\frac{1}{t} \left(\frac{1}{32} \frac{1}{t_1^2} \right) + t \left(\frac{1}{32} \frac{1}{t_1^4} - \frac{7}{96} \frac{1}{t_1^2} \right) + t^3 \left(\frac{1}{32} \frac{1}{t_1^6} - \frac{7}{96} \frac{1}{t_1^4} + \frac{71}{1440} \frac{1}{t_1^2} \right) \right. \\ \left. + t^5 \left(\frac{1}{32} \frac{1}{t_1^8} - \frac{7}{96} \frac{1}{t_1^6} + \frac{71}{1440} \frac{1}{t_1^4} - \frac{191}{30240} \frac{1}{t_1^2} \right) \right. \\ \left. + t^7 \left(\frac{1}{32} \frac{1}{t_1^{10}} - \frac{7}{96} \frac{1}{t_1^8} + \frac{71}{1440} \frac{1}{t_1^6} - \frac{191}{30240} \frac{1}{t_1^4} - \frac{23}{28350} \frac{1}{t_1^2} \right) \right. \\ \left. + t^9 \left(\frac{1}{32} \frac{1}{t_1^{12}} - \frac{7}{96} \frac{1}{t_1^{10}} + \frac{71}{1440} \frac{1}{t_1^8} - \frac{191}{30240} \frac{1}{t_1^6} - \frac{23}{28350} \frac{1}{t_1^4} - \frac{233}{935550} \frac{1}{t_1^2} \right) + \dots \right] \frac{1}{dt} \cdot dt_1.$$

Similarly, around $t = \infty$ we have

$$(7.38) \quad K^{\mathbb{P}^1}(t, t_1) = \left[-t^3 \frac{1}{32} + t \left(-\frac{1}{32} t_1^2 + \frac{7}{96} \right) + \frac{1}{t} \left(-\frac{1}{32} t_1^4 + \frac{7}{96} t_1^2 - \frac{71}{1440} \right) \right. \\ \left. + \frac{1}{t^3} \left(-\frac{1}{32} t_1^6 + \frac{7}{96} t_1^4 - \frac{71}{1440} t_1^2 + \frac{191}{30240} \right) \right. \\ \left. + \frac{1}{t^5} \left(-\frac{1}{32} t_1^8 + \frac{7}{96} t_1^6 - \frac{71}{1440} t_1^4 + \frac{191}{30240} t_1^2 + \frac{23}{28350} \right) \right. \\ \left. + \frac{1}{t^7} \left(-\frac{1}{32} t_1^{10} + \frac{7}{96} t_1^8 - \frac{71}{1440} t_1^6 + \frac{191}{30240} t_1^4 + \frac{23}{28350} t_1^2 + \frac{233}{935550} \right) + \dots \right] \frac{1}{dt} \cdot dt_1.$$

Theorem 7.9. *The Eynard-Orantin differential form $W_{g,n}^{\mathbb{P}^1}(t_1, \dots, t_n)$ is a Laurent polynomial in $t_1^2, t_2^2, \dots, t_n^2$ of degree $2(3g - 3 + n)$ in the stable range $2g - 2 + n > 0$. It satisfies the reciprocity property*

$$(7.39) \quad W_{g,n}^{\mathbb{P}^1}(1/t_1, \dots, 1/t_n) = (-1)^n W_{g,n}^{\mathbb{P}^1}(t_1, \dots, t_n)$$

as a meromorphic symmetric n -form. The highest degree terms form a homogeneous polynomial of degree $2(3g - 3 + n)$, which is given by

$$(7.40) \quad \widehat{W}_{g,n}^{\mathbb{P}^1}(t_1, \dots, t_n) = \frac{(-1)^n}{2^{2g-2+2n}} \sum_{k_1, \dots, k_n \geq 0} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{g,n} \prod_{i=1}^n \left[(2k_i + 1)!! \left(\frac{t_i}{2} \right)^{2k_i} dt_i \right].$$

Indeed it is the same as the generating function of the ψ -class intersection numbers (5.4).

Proof. The statement is proved by induction on $2g - 2 + n$ using the recursion (7.35). The initial cases $(g, n) = (1, 1)$ and $(g, n) = (0, 3)$ are easily verified from the concrete calculations below. Since we are expanding $\frac{1}{t+t_1} + \frac{1}{t-t_1}$ around $t = 0$ and $t = \infty$, it is obvious that the recursion produces a Laurent polynomial in $t_1^2, t_2^2, \dots, t_n^2$ as the result.

The expression of (7.38) tells us that the residue calculation at infinity increases the degree by 4. This is because the leading term of the coefficient of $t^{-(2n+1)}$ is t_1^{2n+4} , and the residue calculation picks up the term t^{2n} . By the induction hypothesis, the right-hand side of (7.35) without the kernel term has homogenous degree $2(3g - 3 + n) - 4$. The reciprocity property also follows by induction using (7.34).

The leading terms of $W_{g,n}^{\mathbb{P}^1}(t_1, \dots, t_n)$ satisfy a topological recursion themselves. We can extract the terms in the kernel that produce the leading terms of the differential forms from (7.36) or (7.38). The result is

$$(7.41) \quad K^{\text{WK}}(t, t_1) = -\frac{1}{32} t^3 \sum_{k=0}^{\infty} \frac{t_1^{2n}}{t^{2n}} \frac{1}{dt} \cdot dt_1 = -\frac{1}{2} \left(\frac{1}{t-t_1} + \frac{1}{t+t_1} \right) \frac{1}{32} t^4 \cdot \frac{1}{dt} \cdot dt_1,$$

which is identical to [10, Theorem 7.4], and also to (5.10). Since the topological recursion uniquely determines all the differential forms from the initial condition, and again since the $(g, n) = (0, 3)$ and $(1, 1)$ cases satisfy (7.40), by induction we obtain (7.40) for all stable values of (g, n) . \square

The $(g, n) = (1, 1)$ Eynard-Orantin differential form is computed using (2.4).

$$(7.42) \quad \begin{aligned} W_{1,1}^{\mathbb{P}^1}(t_1) &= \frac{1}{2\pi i} \int_{\gamma} K^{\mathbb{P}^1}(t, t_1) \left[W_{0,2}^{\mathbb{P}^1}(t, -t) + \frac{dx \cdot dx_1}{(x-x_1)^2} \right] = -\frac{1}{2\pi i} \int_{\gamma} K^{\mathbb{P}^1}(t, t_1) \frac{dt \cdot dt}{4t^2} \\ &= -\frac{1}{64} \left(\frac{1}{2\pi i} \int_{\gamma} \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{1}{\log\left(\frac{(t-1)^2}{(t+1)^2}\right)} \frac{(t^2-1)^2}{t^3} dt \right) dt_1 \\ &= \left(-\frac{1}{128} t_1^2 + \frac{7}{384} + \frac{7}{384} \frac{1}{t_1^2} - \frac{1}{128} \frac{1}{t_1^4} \right) dt_1. \end{aligned}$$

This is in agreement of $W_{1,1}^{\mathbb{P}^1}(t_1) = dF_{1,1}^{\mathbb{P}^1}(t_1)$ and (7.30). From (7.35) we have

$$(7.43) \quad \begin{aligned} W_{0,3}^{\mathbb{P}^1}(t_1, t_2, t_3) &= \frac{1}{2\pi i} \int_{\gamma} K^{\mathbb{P}^1}(t, t_1) \left[W_{0,2}^{\mathbb{P}^1}(t, t_2) W_{0,2}^{\mathbb{P}^1}(-t, t_3) + W_{0,2}^{\mathbb{P}^1}(t, t_3) W_{0,2}^{\mathbb{P}^1}(-t, t_2) \right] \\ &= -\frac{1}{16} \left(1 + \frac{1}{t_1^2 t_2^2 t_3^2} \right) dt_1 dt_2 dt_3. \end{aligned}$$

It is also in agreement with (7.28).

Norbury and Scott conjecture the following

Conjecture 7.10 (Norbury-Scott Conjecture [62]). *For (g, n) in the stable range we have*

$$(7.44) \quad W_{g,n}^{\mathbb{P}^1}(t_1, \dots, t_n) = d_1 \cdots d_n F_{g,n}^{\mathbb{P}^1}(t_1, \dots, t_n).$$

The conjecture is verified for $g = 0$ and $g = 1$ cases in [62]. We recall that the Eynard-Orantin recursion for simple Hurwitz numbers is essentially the Laplace transform of the cut-and-join equation [24]. For the case of the counting problem of clean Belyi morphisms the recursion is the Laplace transform of the edge-contraction operation of Theorem 3.3.

Question 7.11. *What is the equation among the stationary Gromov-Witten invariants of \mathbb{P}^1 whose Laplace transform is the Eynard-Orantin recursion (7.35)?*

APPENDIX A. CALCULATION OF THE LAPLACE TRANSFORM

In this appendix we give the proof of Theorem 4.3.

Proposition A.1. *Let us use the x_j -variables defined by $x_j = e^{w_j}$, and write*

$$W_{g,n}^D(t_1, \dots, t_n) = w_{g,n}(x_1, \dots, x_n) dx_1 \cdots dx_n.$$

Then the Laplace transform of the recursion formula (3.15) is the following differential recursion:

$$(A.1) \quad -x_1 w_{g,n}(x_1, \dots, x_n) \\ = \sum_{j=2}^n \frac{\partial}{\partial x_j} \left(\frac{1}{x_j - x_1} (w_{g,n-1}(x_2, \dots, x_n) - w_{g,n-1}(x_1, x_2, \dots, \widehat{x}_j, \dots, x_n)) \right) \\ + w_{g-1,n+1}(x_1, x_1, x_2, \dots, x_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2, \dots, n\}}} w_{g_1, |I|+1}(x_1, x_I) w_{g_2, |J|+1}(x_1, x_J).$$

Proof. The operation we wish to do is to apply

$$(-1)^n \sum_{\mu_1, \dots, \mu_n > 0} \mu_2 \cdots \mu_n \prod_{i=1}^n \frac{1}{x_i^{\mu_i+1}}$$

to each side of (3.15). Then by (4.13), the left-hand side becomes $w_{g,n}(x_1, \dots, x_n)$.

The second line of (3.15) is straightforward. Let us just consider the first term, since the computation of the second term is the same.

$$(-1)^n \sum_{\mu_1, \dots, \mu_n > 0} \mu_2 \cdots \mu_n \prod_{i=1}^n \frac{1}{x_i^{\mu_i+1}} \sum_{\alpha+\beta=\mu_1-2} \alpha \beta D_{g-1,n+1}(\alpha, \beta, \mu_2, \dots, \mu_n) \\ = -\frac{1}{x_1} (-1)^{n+1} \sum_{\mu_2, \dots, \mu_n > 0} \sum_{\alpha, \beta > 0} \alpha \beta \mu_2 \cdots \mu_n D_{g-1,n+1}(\alpha, \beta, \mu_2, \dots, \mu_n) \frac{1}{x_1^{\alpha+1}} \cdot \frac{1}{x_1^{\beta+1}} \prod_{i=2}^n \frac{1}{x_i^{\mu_i+1}} \\ = -\frac{1}{x_1} w_{g-1,n+1}(x_1, x_1, x_2, \dots, x_n).$$

Thus the second line of (3.15) produces

$$-\frac{1}{x_1} \left(w_{g-1,n+1}(x_1, x_1, x_2, \dots, x_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2, \dots, n\}}} w_{g_1, |I|+1}(x_1, x_I) w_{g_2, |J|+1}(x_1, x_J) \right).$$

To calculate the operation on the first line of (3.15), let us fix $j > 1$ and set $\nu = \mu_1 + \mu_j - 2 \geq 0$. Then

$$(A.2) \quad (-1)^n \sum_{\mu_1, \dots, \mu_n > 0} \mu_2 \cdots \mu_n (\mu_1 + \mu_j - 2) \\ \times D_{g,n-1}(\mu_1 + \mu_j - 2, \mu_2, \dots, \widehat{\mu}_j, \dots, \mu_n) \prod_{i=1}^n \frac{1}{x_i^{\mu_i+1}} \\ = -\sum_{\nu=0}^{\infty} \sum_{\mu_2, \dots, \widehat{\mu}_j, \dots, \mu_n > 0} (-1)^{n-1} \nu \mu_2 \cdots \widehat{\mu}_j \cdots \mu_n$$

$$\times D_{g,n-1}(\nu, \mu_2, \dots, \widehat{\mu}_j, \dots, \mu_n) \frac{1}{x_1^{\nu+1}} \prod_{i \neq 1, j} \frac{1}{x_i^{\mu_i+1}} \sum_{\mu_j=1}^{\nu+1} \mu_j x_1^{\mu_j-2} \frac{1}{x_j^{\mu_j+1}}.$$

Assuming $|x_1| < |x_j|$, we calculate

$$\begin{aligned} \text{(A.3)} \quad \sum_{\mu_j=1}^{\nu+1} \mu_j x_1^{\mu_j-2} \frac{1}{x_j^{\mu_j+1}} &= -\frac{1}{x_1^2} \frac{\partial}{\partial x_j} \sum_{\mu_j=0}^{\nu+1} \left(\frac{x_1}{x_j} \right)^{\mu_j} = -\frac{1}{x_1^2} \frac{\partial}{\partial x_j} \left(\frac{1}{1 - \frac{x_1}{x_j}} - \frac{\left(\frac{x_1}{x_j} \right)^{\nu+2}}{1 - \frac{x_1}{x_j}} \right) \\ &= -\frac{1}{x_1^2} \frac{\partial}{\partial x_j} \left(\frac{1}{1 - \frac{x_1}{x_j}} \right) + x_1^\nu \frac{\partial}{\partial x_j} \left(\frac{1}{x_j - x_1} \frac{1}{x_j^{\nu+1}} \right). \end{aligned}$$

We then substitute (A.3) in (A.2) and obtain

$$\begin{aligned} \text{(A.4)} \quad \text{(A.2)} &= w_{g,n-1}(x_1, x_2, \dots, \widehat{x}_j, \dots, x_n) \frac{1}{x_1^2} \frac{\partial}{\partial x_j} \left(\frac{1}{1 - \frac{x_1}{x_j}} \right) \\ &\quad - \frac{1}{x_1} \frac{\partial}{\partial x_j} \left(\frac{1}{x_j - x_1} w_{g,n-1}(x_2, \dots, x_j, \dots, x_n) \right) \\ &= -\frac{1}{x_1} \frac{\partial}{\partial x_j} \left(\frac{1}{x_j - x_1} (w_{g,n-1}(x_2, \dots, x_j, \dots, x_n) - w_{g,n-1}(x_1, x_2, \dots, \widehat{x}_j, \dots, x_n)) \right). \end{aligned}$$

This completes the proof. \square

Proof of Theorem 4.3. When the curve is split into two pieces, the second term of the third line of (A.1) contains contributions from unstable geometries $(g, n) = (0, 1)$ and $(0, 2)$. We first separate them out. For $g_1 = 0$ and $I = \emptyset$, or $g_2 = 0$ and $J = \emptyset$, we have a contribution of

$$2w_{0,1}(x_1)w_{g,n}(x_1, x_2, \dots, x_n).$$

Similarly, for $g_1 = 0$ and $I = \{j\}$, or $g_2 = 0$ and $J = \{j\}$, we have

$$2 \sum_{j=2}^n w_{0,2}(x_1, x_j) w_{g,n-1}(x_1, \dots, \widehat{x}_j, \dots, x_n).$$

Since $W_{0,1}^D$ and $W_{0,2}^D$ are defined on the spectral curve, it is time for us to switch to the preferred coordinate t of (4.1) now. We thus introduce

$$\text{(A.5)} \quad W_{g,n}^D(t_1, \dots, t_n) = w_{g,n}^D(t_1, \dots, t_n) dt_1 \cdots dt_n = w_{g,n}(x_1, \dots, x_n) dx_1 \cdots dx_n.$$

Since $w_{0,1}(x) = -z(x)$, we have

$$\begin{aligned} w_{0,1}(x) &= -\frac{t+1}{t-1} \\ w_{0,2}(x_1, x_2) &= \frac{1}{(t_1+t_2)^2} \frac{(t_1^2-1)^2}{8t_1} \frac{(t_2^2-1)^2}{8t_2} \\ w_{g,n}(x_1, \dots, x_n) &= (-1)^n w_{g,n}^D(t_1, \dots, t_n) \prod_{i=1}^n \frac{(t_i^2-1)^2}{8t_i}. \end{aligned}$$

Thus (A.1) is equivalent to

$$2 \left(\frac{t_1^2+1}{t_1^2-1} - \frac{t_1+1}{t_1-1} \right) w_{g,n}^D(t_1, \dots, t_n)$$

$$\begin{aligned}
&= \sum_{j=2}^n \left(\frac{(t_1^2 - 1)(t_j^2 - 1)^2}{16(t_1^2 - t_j^2)^2} \frac{8t_j}{(t_j^2 - 1)^2} w_{g,n-1}^D(t_1, \dots, \hat{t}_j, \dots, t_n) \right. \\
&\quad \left. + \frac{\partial}{\partial t_j} \left(\frac{(t_1^2 - 1)(t_j^2 - 1)}{4(t_1^2 - t_j^2)} \frac{8t_1}{(t_1^2 - 1)^2} \frac{(t_j^2 - 1)^2}{8t_j} w_{g,n-1}^D(t_2, \dots, t_n) \right) \right) \\
&\quad + \frac{(t_1^2 - 1)^2}{8t_1} \left(w_{g-1,n+1}^D(t_1, t_1, t_2, \dots, t_n) + \sum_{\substack{\text{stable} \\ I \sqcup J = \{2, \dots, n\}}}^{g_1 + g_2 = g} w_{g_1, |I|+1}^D(t_1, t_I) w_{g_2, |J|+1}^D(t_1, t_J) \right) \\
&\quad + 2 \sum_{j=2}^n \frac{1}{(t_1 + t_j)^2} \frac{(t_1^2 - 1)^2}{8t_1} w_{g,n-1}^D(t_1, \dots, \hat{t}_j, \dots, t_n) \\
&= \sum_{j=2}^n \left(\left(\frac{t_j(t_1^2 - 1)^2}{2(t_1^2 - t_j^2)^2} + \frac{1}{(t_1 + t_j)^2} \frac{(t_1^2 - 1)^2}{4t_1} \right) w_{g,n-1}^D(t_1, \dots, \hat{t}_j, \dots, t_n) \right. \\
&\quad \left. + \frac{t_1}{t_1^2 - 1} \frac{\partial}{\partial t_j} \left(\frac{(t_j^2 - 1)^3}{4t_j(t_1^2 - t_j^2)} w_{g,n-1}^D(t_2, \dots, t_n) \right) \right) \\
&\quad + \frac{(t_1^2 - 1)^2}{8t_1} \left(w_{g-1,n+1}^D(t_1, t_1, t_2, \dots, t_n) + \sum_{\substack{\text{stable} \\ I \sqcup J = \{2, \dots, n\}}}^{g_1 + g_2 = g} w_{g_1, |I|+1}^D(t_1, t_I) w_{g_2, |J|+1}^D(t_1, t_J) \right).
\end{aligned}$$

Since

$$2 \left(\frac{t_1^2 + 1}{t_1^2 - 1} - \frac{t_1 + 1}{t_1 - 1} \right) = -\frac{4t_1}{t_1^2 - 1},$$

we obtain

$$\begin{aligned}
\text{(A.6)} \quad w_{g,n}^D(t_1, \dots, t_n) &= - \sum_{j=2}^n \left(\frac{\partial}{\partial t_j} \left(\frac{(t_j^2 - 1)^3}{16t_j(t_1^2 - t_j^2)} w_{g,n-1}^D(t_2, \dots, t_n) \right) \right. \\
&\quad \left. + \frac{(t_1^2 - 1)^3}{16t_1^2} \frac{t_1^2 + t_j^2}{(t_1^2 - t_j^2)^2} w_{g,n-1}^D(t_1, \dots, \hat{t}_j, \dots, t_n) \right) \\
&\quad - \frac{(t_1^2 - 1)^3}{32t_1^2} \left(w_{g-1,n+1}^D(t_1, t_1, t_2, \dots, t_n) + \sum_{\substack{\text{stable} \\ I \sqcup J = \{2, \dots, n\}}}^{g_1 + g_2 = g} w_{g_1, |I|+1}^D(t_1, t_I) w_{g_2, |J|+1}^D(t_1, t_J) \right).
\end{aligned}$$

Now let us compute the integral

$$\begin{aligned}
\text{(A.7)} \quad W_{g,n}^D(t_1, \dots, t_n) &= -\frac{1}{64} \frac{1}{2\pi i} \int_{\gamma} \left(\frac{1}{t + t_1} + \frac{1}{t - t_1} \right) \frac{(t^2 - 1)^3}{t^2} \cdot \frac{1}{dt} \cdot dt_1 \\
&\quad \times \left[\sum_{j=2}^n \left(W_{0,2}^D(t, t_j) W_{g,n-1}(-t, t_2, \dots, \hat{t}_j, \dots, t_n) + W_{0,2}^D(-t, t_j) W_{g,n-1}(t, t_2, \dots, \hat{t}_j, \dots, t_n) \right) \right]
\end{aligned}$$

$$+ W_{g-1, n+1}^D(t, -t, t_2, \dots, t_n) + \left. \sum_{\substack{\text{stable} \\ g_1+g_2=g \\ I \sqcup J = \{2, 3, \dots, n\}}} W_{g_1, |I|+1}^D(t, t_I) W_{g_2, |J|+1}^D(-t, t_J) \right].$$

Recall that For $2g-2+n > 0$, $w_{g,n}^D(t_1, \dots, t_n)$ is a Laurent polynomial in t_1^2, \dots, t_n^2 . Thus the third line of (A.7) is immediately calculated because the integration contour γ of Figure 4.1 encloses $\pm t_1$ and contributes residues with the negative sign. The result is exactly the last line of (A.6). Similarly, since

$$\begin{aligned} & W_{0,2}^D(t, t_j) W_{g,n-1}(-t, t_2, \dots, \hat{t}_j, \dots, t_n) + W_{0,2}^D(-t, t_j) W_{g,n-1}(t, t_2, \dots, \hat{t}_j, \dots, t_n) \\ &= - \left(\frac{1}{(t+t_j)^2} + \frac{1}{(t-t_j)^2} \right) w_{g,n-1}^D(t, t_2, \dots, \hat{t}_j, \dots, t_n) dt dt dt_2 \cdots \widehat{dt}_j \cdots dt_n, \end{aligned}$$

the residues at $\pm t_1$ contributes

$$- \frac{(t_1^2 - 1)^3 (t_1^2 + t_j^2)}{16 t_1^2 (t_1^2 - t_j^2)^2} w_{g,n-1}^D(t_1, \dots, \hat{t}_j, \dots, t_n).$$

This is the same as the second line of the right-hand side of (A.6).

Within the contour γ , there are second order poles at $\pm t_j$ for each $j \geq 2$ that come from $W_{0,2}^D(\pm t, t_j)$. Note that $W_{0,2}^D(t, t_j)$ acts as the Cauchy differentiation kernel. We calculate

$$\begin{aligned} & \frac{1}{64} \frac{1}{2\pi i} \int_{\gamma} \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \sum_{j=2}^n \left(w_{0,2}^D(t, t_j) w_{g,n-1}^D(-t, t_2, \dots, \hat{t}_j, \dots, t_n) \right. \\ & \quad \left. + w_{0,2}^D(-t, t_j) w_{g,n-1}^D(t, t_2, \dots, \hat{t}_j, \dots, t_n) \right) \\ &= - \frac{1}{32} \frac{\partial}{\partial t_j} \left(\left(\frac{1}{t_j+t_1} + \frac{1}{t_j-t_1} \right) \frac{(t_j^2-1)^3}{t_j^2} w_{g,n-1}^D(t_j, t_2, \dots, \hat{t}_j, \dots, t_n) \right) \\ &= - \frac{1}{16} \frac{\partial}{\partial t_j} \left(\frac{1}{t_j^2-t_1^2} \frac{(t_j^2-1)^3}{t_j} w_{g,n-1}^D(t_j, t_2, \dots, \hat{t}_j, \dots, t_n) \right). \end{aligned}$$

This gives the first line of the right-hand side of (A.6). We have thus completes the proof of Theorem 4.3. \square

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