82. *Algebraic Geometry of Soliton Equations**)

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The purpose of this paper is to classify all the subdynamical systems of the $K-P$ dynamical system $(\hat{G}, T)$ defined in [2] in terms of commutative algebras. We show that every orbit in $(\hat{G}, T)$ is locally isomorphic to a certain first cohomology group $H^1(A)$ associated with a commutative algebra A and the $K-P$ dynamical system is nothing but a dynamical system of a linear motion on this cohomology group. In the case of so called quasi-periodic solutions, it is known that the $K-P$ dynamical system determines a linear motion on the Jacobian varieties of algebraic curves. Our results are the widest extension of this classical result. We also characterize all the finite dimensional orbits in $(\hat{G}, T)$. We show that an orbit is of finite dimension if and only if our cohomology group $H^1(A)$ is isomorphic to $H^1(C, \mathcal{O}_C)$ for a certain complete algebraic curve C defined over the complex number field $\boldsymbol{C}$. This enables us to solve the *Schottky problem* in the following manner; an Abelian variety is a Jacobian variety if and only if it appears as an orbit in $(\hat{G}, T)$ (cf. [4]).

In this paper we use notations defined in [1] and [2] freely.

1. Subdynamical systems of $(\hat{G}, T)$ and commutative algebras. Let $H=\boldsymbol{C}((\partial^{-1}))$. This is a maximal commutative subalgebra in the Lie algebra E of [1]. Let $\mathcal{A}=\{A\subset H \mid A$ is a $\boldsymbol{C}$-subalgebra with unity and $A\cap \boldsymbol{C}[[\partial^{-1}]]\cdot\partial^{-1}=0\}$. Define $X_A=\{S\in G \mid SAS^{-1}\subset D\}$ and $\hat{X}_A=\{S\partial S^{-1} \mid S\in X_A\}$. The condition $A\cap \boldsymbol{C}[[\partial^{-1}]]\cdot\partial^{-1}=0$ intends to avoid the trivial case $X_A=\phi$. Also by this condition A has transcendence degree 1 over $\boldsymbol{C}$. Mikio Sato has originally introduced the notion of A to study several orbits.

Proposition 1.1. *X_A is a time invariant subspace in G. So $(\hat{X}_A, T)$ is a subdynamical system of $(\hat{G}, T)$.*

Proof. For every $S\in X_A$ we have a unique solution $S(t)$ to the Sato equation starts at $S(0)=S$ ([1]). So it is sufficient to prove $\partial/\partial t_n(S(t)AS(t)^{-1})\subset D$ for every $n\geqslant 1$. Define

$$L=S(t)\partial S(t)^{-1},\quad Z=\sum_{n=1}^{\infty}(L^n)_+dt_n \quad\text{and}\quad Z^c=-\sum_{n=1}^{\infty}(L^n)_-dt_n.$$

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Then for every $a=a(\partial)\in A$, we have

$$d(S(t)aS(t)^{-1})=[Z^c, a(L)]=[Z, a(L)].$$

Since $a(L)\in D$ and Z has coefficients in D, we conclude that $\partial/\partial t_n(S(t)aS(t)^{-1})\in D$. Q. E. D.

We call $(\hat{X}_A, T)$ the *subdynamical system associated with* $A\in\mathcal{A}$. Note that if A and $A'\in\mathcal{A}$ satisfy $A\subset A'$, then $\hat{X}_{A'}\subset\hat{X}_A$. So $\mathcal{X}=\{\hat{X}_A|$ subdynamical system associated with $A\in\mathcal{A}\}$ is a dual lattice of $\mathcal{A}$ with respect to the inclusion relation. We call an orbit in $\hat{X}_A$ is an *A-maximal orbit* if it is not contained in $\hat{X}_{A'}$ for some $A'\supset A$.

In the rest of this section we study the structure of A-maximal orbits.

Definition 1.2. The cohomology group $H^1(A)$ of an algebra $A\in\mathcal{A}$ is defined by the first cohomology group of the complex

$$0\longrightarrow A\oplus \boldsymbol{C}[[\partial^{-1}]]\cdot\partial^{-1}\longrightarrow \boldsymbol{C}((\partial^{-1}))\longrightarrow 0.$$

Theorem 1.3. *Every A-maximal orbit M_A in $\hat{X}_A$ is locally isomorphic to $H^1(A)$ and the K−P dynamical system restricted to M_A is just a dynamical system of a linear motion with respect to the linear structure of $H^1(A)$.*

Proof. Start with a point $L\in M_A$ and let S be the Sato operator of L. First we note that every $\partial^n\in\boldsymbol{C}[\partial]$ corresponds to a different time evolution by the Lax equation $\partial L/\partial t_n=[(S\partial^n S^{-1})_+, L]$. Since every element $a\in A$ corresponds to a stationary time because of the fact $[(SaS^{-1})_+, L]=0$, we see that the cohomology group

$$H^1(A)=\boldsymbol{C}((\partial^{-1}))/(A\oplus\boldsymbol{C}[[\partial^{-1}]]\cdot\partial^{-1})\cong\boldsymbol{C}[\partial]\cdot\partial/(A/\boldsymbol{C}[[\partial^{-1}]])$$

corresponds to the essential time evolutions of L. We can take finitely or infinitely many elements $b_1, b_2, \cdots\in\boldsymbol{C}[\partial]\cdot\partial$ as a basis of $H^1(A)$. Let $b_i=\sum_{j=1}^{n_i} b_{ij}\partial^j$ and define $\partial/\partial s_i=\sum_j b_{ij}(\partial/\partial t_j)$. Then the equations $\partial L/\partial s_i=[(Sb_iS^{-1})_+, L]$ determine the essential time evolutions, hence the map $H^1(A)\ni b_i\mapsto\partial L/\partial s_i\in T_L(M_A)$ is isomorphic, where $T_L(M_A)$ denotes the tangent space of M_A at L. Since this isomorphism does not depend on $L\in M_A$, we conclude that M_A is locally isomorphic to $H^1(A)$ and we can take $s_1, s_2, \cdots$ as a local coordinate system of M_A, where s_i's are linear combinations of t_j's satisfying the conditions $\partial/\partial s_i=\sum_j b_{ij}(\partial/\partial t_j)$. Thus the time evolution restricted to M_A determines just a linear motion. Q. E. D.

The corresponding time evolution operator U defined in [1] is given by $U=e^{s_1b_1+s_2b_2+\cdots}$.

Remark. It is known that the subdynamical system $(\hat{X}_{C[\partial^2]}, T)$ corresponds to the hierarchy of the Korteweg de Vries equation and $(\hat{X}_{C[\partial^3]}, T)$ to the hierarchy of the Boussinesq equation. In this way every element $A\in\mathcal{A}$ corresponds to a system of non-linear partial differential equations which defines the subdynamical system $(\hat{X}_A, T)$.

2. Orbits of the $K-P$ dynamical system. Here we study the opposite direction. We construct $A \in \mathcal{A}$ from an orbit M in $(\hat{G}, T)$. Let L be the solution to the Lax equation corresponding to M and S be the Sato operator of L. L defines a homomorphism $\ell : T_0(T) \ni \partial/\partial t_n \mapsto \partial L/\partial t_n \in T_L(M)$ between the tangent spaces.

Lemma 2.1. *If* $\mathrm{Ker}(\ell) \neq 0$, *then* $\dim \mathrm{Ker}(\ell) = \infty$.

Proof. Take an element $\partial/\partial s \in \mathrm{Ker}(\ell)$. Since T is the inductive limit space, $\partial/\partial s = \sum_{j=1}^{k} c_j(\partial/\partial t_j)$ for some constants $c_1, c_2, \cdots, c_k$. Let $B_s = \sum_j c_j(L^j)_+$. Since $[B_s, L] = \partial L/\partial s = 0$, we have $[B_s^n, L] = 0$ for every $n \geqslant 1$. Then, as we have studied in §1 of [1], (B_s^n, L) is a Lax pair, hence B_s^n can be written as a linear combination of $(L^j)_+$'s like $B_s^n = \sum_j e_j(L^j)_+$. Let $\partial/\partial s_n = \sum_j e_j(\partial/\partial t_j)$. Then the Lax equation says that $\partial/\partial s_n \in \mathrm{Ker}(\ell)$ because $\partial L/\partial s_n = [B_s^n, L] = 0$. Since $(\partial/\partial s_n)$'s are linearly independent, we conclude $\dim \mathrm{Ker}(\ell) = \infty$. Q. E. D.

Let $\partial/\partial s_1, \partial/\partial s_2, \cdots$ be a basis of $\mathrm{Ker}(\ell)$. Let $\partial/\partial s_j = \sum_{j=1}^{n_i} c_{ij}(\partial/\partial t_j)$ and $B_{s_i} = \sum_j c_{ij}(L^j)_+$. Because of the compatibility of $(\partial/\partial t_j)$'s, we can take c_{ij} to be constant. Since $\partial L/\partial s_i = 0$, $\partial B_{s_j}/\partial s_i = 0$ for every i and j. Then the Zakharov-Shabat equation says that $[B_{s_i}, B_{s_j}] = \partial B_{s_j}/\partial s_i - \partial B_{s_i}/\partial s_j = 0$. Define $A = S^{-1}C[B_{s_1}, B_{s_2}, \cdots]S \subset \mathcal{E}_0$. This is a commutative subalgebra in $\mathcal{E}_0$.

Proposition 2.2. *The algebra A is an element of $\mathcal{A}$.*

Proof. It is obvious that $A \cap C[[\partial^{-1}]]\cdot\partial^{-1} = 0$ because $SAS^{-1} \subset \mathcal{D}_0$. Let us prove $S^{-1}B_{s_i}S \in H$ for every i. By the Sato equation $\partial S/\partial t_j = -(L^j)_- S$ we see that $S^{-1}B_{s_i}S = S^{-1}\sum_j c_{ij}[L^j - (L^j)_-]S = \sum_j c_{ij}\partial^j + S^{-1}(\partial S/\partial s_i)$. First we know that $S^{-1}(\partial S/\partial s_i)$ does not depend on x because $[S^{-1}(\partial S/\partial s_i), \partial] = S^{-1}(\partial L/\partial s_i)S = 0$. We also see that

$$\begin{aligned}\frac{\partial}{\partial t_n}\left(S^{-1}\frac{\partial S}{\partial s_i}\right) &= -S^{-1}\frac{\partial S}{\partial t_n}S^{-1}\frac{\partial S}{\partial s_i} + S^{-1}\frac{\partial}{\partial t_n}\left(\frac{\partial S}{\partial s_i}\right)\\ &= S^{-1}(L^n)_-\frac{\partial S}{\partial s_i} - S^{-1}\frac{\partial}{\partial s_i}[(L^n)_- S] = 0,\end{aligned}$$

thus we have $S^{-1}B_{s_i}S \in H$. Q. E. D.

Our M is an A-maximal orbit in the subdynamical system $(\hat{X}_A, T)$ associated with A. Thus we have obtained the following:

Theorem 2.3. *For every orbit M in $(\hat{G}, T)$ there exists a unique element $A \in \mathcal{A}$ and a subdynamical system $(\hat{X}_A, T)$ associated with A such that $\hat{X}_A$ contains M as an A-maximal orbit.*

3. Why do curves and their Jacobians arise as solutions to the soliton equations? Because we have following theorem.

Theorem 3.1. *An orbit M in $(\hat{G}, T)$ is of finite dimension if and only if it comes from a complate algebraic curve C (might be singular) and M is locally isomorphic to the connected component* $\mathrm{Pic}^0(C)$ *of the Picard group of C.*

Proof. Take $A\in\mathcal{A}$ corresponds to M as given in § 2. Since $A\subset E$, we can define the order of elements in A. Let $A_n=\{a\in A|$ $\mathrm{ord}(a)\leqslant n\}$. Note that $A_{-1}=0$. Now define the graded algebra $gr(A)$ $=\sum_{n=0}^{\infty}A_n$. Since M is locally isomorphic to $H^1(A)$,

$$\dim \boldsymbol{C}[\partial]\cdot\partial/(A/\boldsymbol{C}[[\partial^{-1}]])<\infty.$$

We can conclude that $gr(A)$ has transcendence degree 2 over $\boldsymbol{C}$ by using Mumford's observation [3]. Let $C=\mathrm{Proj}(gr(A))$. Mumford has studied in [3] that C is a one point compactification of an affine algebraic curve $C-p=\mathrm{Spec}(A)$, where p is a regular point in C. Let $\mathcal{U}$ be a Stein neighborhood of p in C and λ^{-1} be a local coordinate of $\mathcal{U}$ such that $\lambda^{-1}=0$ at p. Since $H^1(C-p, \mathcal{O}_C)=0$ and $H^1(\mathcal{U}, \mathcal{O}_C)=0$, we can calculate

$$\begin{aligned} H^1(C, \mathcal{O}_C)&=\Gamma(\mathcal{U}-p, \mathcal{O}_C)/[\Gamma(C-p, \mathcal{O}_C)+\Gamma(\mathcal{U}, \mathcal{O}_C)] \\ &\cong \boldsymbol{C}((\lambda^{-1}))/(A\oplus\boldsymbol{C}[[\lambda^{-1}]]\cdot\lambda^{-1})=H^1(A), \end{aligned}$$

where we have identified ∂ with λ. Thus we conclude that M is locally isomorphic to $\mathrm{Pic}^0(C)=H^1(C, \mathcal{O}_C)/H^1(C, \boldsymbol{Z})$ and the soliton equations define (at least locally) a linear motion on $\mathrm{Pic}^0(C)$.

Conversely suppose we have a complete algebraic curve C defined over $\boldsymbol{C}$. Take a non-singular point $p\in C$ and a local parameter λ^{-1} at p such that $\lambda^{-1}=0$ defines p. By expanding in λ we have an injection $\Gamma(C-p, \mathcal{O}_C)\hookrightarrow\boldsymbol{C}((\lambda^{-1}))$. Let A be the image. Then by identifying λ with ∂, we obtain M_A of finite dimension corresponding to A as studied in § 1. Q. E. D.

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