

- 1 (32 pts.)** Determine whether or not the following limits exist, and calculate them. If the limit does not exist as a number, state whether or not it can be written as  $\infty$  or  $-\infty$ .

(a)  $\lim_{x \rightarrow 3} \frac{x-3}{x^2-2x-3}$

**Answer:**  $\lim_{x \rightarrow 3} \frac{x-3}{x^2-2x-3} = 1/4$

(b)  $\lim_{t \rightarrow 0} \frac{\sqrt{t+1}-1}{t}$

**Answer:**  $\lim_{t \rightarrow 0} \frac{\sqrt{t+1}-1}{t} = 1/2$

(c)  $\lim_{x \rightarrow 1^+} \frac{2-x^2}{1-x}$

**Answer:**  $\lim_{x \rightarrow 1^+} \frac{2-x^2}{1-x} = -\infty$

(d)  $\lim_{x \rightarrow \infty} \sin x$

**Answer:**  $\lim_{x \rightarrow \infty} \sin x$  does not exist

- 2 (8 pts.)** Using the sandwich theorem, show that  $\lim_{x \rightarrow 0} x^4(1 - \cos x) = 0$ .

**Answer:**  $0 \leq x^4(1 - \cos x) \leq 2x^4$ .

- 3 (12 pts.)** Directly from the definition of a limit, show that  $\lim_{x \rightarrow -3} x^2 = 9$ .

**Answer:** Given  $\epsilon > 0$ , if  $\epsilon < 9$  then we can use

$$\delta = \min\{3 - \sqrt{9 - \epsilon}, \sqrt{9 + \epsilon} - 3\}.$$

(Notice that if you need to know a precise valid formula for  $\delta$ , you might need to figure out which possibility is smaller. But to prove the limit is 9, it is enough to know that a  $\delta$  exists, so taking the minimum of two possibilities is no problem). For  $\epsilon \gg 9$ ,  $\delta$  may be chosen to be 1 (for instance).

4 (16 pts.) Find  $\lim_{x \rightarrow \infty} \frac{1 - \cos(1/x)}{(1/2x)}$ . (Hint: use the double angle formula)

**Answer:**  $\lim_{x \rightarrow \infty} \frac{1 - \cos(1/x)}{(1/2x)} = 0$ , using the trig identity  $\cos(1/x) = 1 - \sin^2(1/2x)$ .

5 (16 pts.) Graph the function

$$f(x) = \begin{cases} \frac{x^3 - 2x^2 + 2x - 4}{x^2 - 3x + 2} : & x \neq 1, 2 \\ 4 : & x = 1, 2 \end{cases}$$

(Hint: factor the denominator first) Find all asymptotes to the graph. At which points is this function continuous, and at which points is it discontinuous? For each discontinuity, say whether or not it is removable.

**Answer:** [Graph omitted] For  $x \neq 1, 2$ ,  $\frac{x^3 - 2x^2 + 2x - 4}{x^2 - 3x + 2} = \frac{x^2 + 2}{x - 1}$ . This has a vertical asymptote at  $x = 1$ , since the numerator is non-zero. At  $x = 2$ , the limit is 6. This means  $f(x)$  is discontinuous at  $x = 1$  and  $x = 2$ , but the discontinuity at  $x = 2$  is removable, while the discontinuity at  $x = 1$  is not removable. Besides the asymptote  $x = 1$ , there is an oblique asymptote:  $\frac{x^2 + 2}{x - 1} = x + 1 + \frac{3}{x - 1}$ , so  $y = x + 1$  is also an asymptote.

6 (8 pts.) Show that the equation  $x^3 - x - 1 = 0$  has a solution in the interval  $[1, 2]$ .

**Answer:** The function  $f(x) = x^3 - x - 1$  is a polynomial, therefore continuous.  $f(1) = -1$ , but  $f(2) = 5$ , so by the intermediate value theorem,  $f(x) = 0$  somewhere in between  $x = 1$  and  $x = 2$ .

7 (8 pts.) At  $t$  seconds after liftoff, the height of a rocket is  $4t^2$  feet. How fast is the rocket climbing 10 seconds after liftoff? (Compute this from our definitions, without using derivative laws)

**Answer:** 80 feet/second.