NONSINGULAR CURVES

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The primary goal of this note is to prove that every abstract nonsingular curve can be realized as an open subset of a (unique) nonsingular projective curve. Note that this encapsulates two facts in one: that every nonsingular abstract curve is quasiprojective, and that it can be "compactified" into a projective curve without introducing singularities.

We start from the definitions and state the necessary background algebra.

1. Curves and their local rings

Our discussion of the abstract definition of a variety allows us to work transparently with abstract curves.

Definition 1.1. A **curve** is a variety of dimension 1.

Before we can say much about curves, we have to introduce some algebra which plays an important role.

Definition 1.2. A **discrete valuation ring** is an integral domain R together with a nonconstant map $\nu : K(R)^* \to \mathbb{Z}$ satisfying the following conditions:

(i) $\nu(xy) = \nu(x) + \nu(y)$;
(ii) $\nu(x + y) \geqslant \min(\nu(x), \nu(y))$ for $x \neq -y$;
(iii) $\nu(x) \geqslant 0$ if and only if $x \in R$.

Remark 1.3. One shows that a discrete valuation ring is a local ring, with principal maximal ideal, and every non-zero ideal simply a multiple of the maximal ideal. The valuation is then determined by which multiple of the maximal ideal any given element is in, and is thus determined entirely by the ring structure. That is to say, the map $\nu : K(R)^* \to \mathbb{Z}$ in the definition is unique, if it exists.

The relevance to the study of curves is the following basic result in commutative algebra:

Proposition 1.4. *Let P be a nonsingular point of a curve C. Then the local ring $\mathcal{O}_{P,C}$ is a discrete valuation ring.*

The intuition is quite simple: (nonsingular) points P of curves have the property that functions regular at P have a well-defined order of vanishing, and that any rational function has a well-defined order of vanishing or pole at P. A rational function is regular at P if and only if the order at P is non-negative. We can express this geometric intuition rather clearly as follows:

Corollary 1.5. *Let P be a nonsingular point of a curve C. Then there exists an open neighborhood U of P and a regular function t on U such that for all $V \subseteq U$ open, and $f \in \mathcal{O}(V)$, we have $f(P) = 0$ if and only if t divides f in $\mathcal{O}(V)$.*

In particular, for all $f \in \mathcal{O}(V \smallsetminus \{P\})$, we can write $f = t^{\nu} g$, where $\nu = \nu(f)$ using the valuation on $K(C)$ coming from $\mathcal{O}_{P,C}$, and $g \in \mathcal{O}(V)$ satisfies $g(P) \neq 0$.

Proof. We first note that for any DVR R, by condition (iii) of the definition the units of R are precisely the set of elements with valuation 0, so the maximal ideal of R is precisely the set of elements with positive valuation.

Now, in our situation choose $t \in \mathcal{O}_{P,C}$ such that $\nu(t)$ is minimal positive, and (by mild abuse of notation) let $\langle U, t\rangle$ be a representative for t. Then $Z(t)$ consists of a finite set of points of U, so by restricting U to a smaller open subset, we may assume that t is nonvanishing away from P on U. Given $V \subseteq U$ and $f \in \mathcal{O}(V)$, clearly if t divides f in $\mathcal{O}(V)$ we have $f(P) = 0$. Conversely, if $f(P) = 0$ then f has positive valuation in $\mathcal{O}_{P,C}$, so by minimality f/t has nonnegative valuation, and is in $\mathcal{O}_{P,C}$. But since $t(Q) \neq 0$ for $Q \neq P$, we have f/t regular on $V \smallsetminus \{P\}$, and because $f/t \in \mathcal{O}_{P,C}$ as well, we conclude f/t is regular on V, proving the first assertion.

The second assertion follows immediately by induction from the first if f is regular at P. In general, let $g = ft^{-\nu}$. Then g is regular on $V \smallsetminus \{P\}$ because t is invertible on $U \smallsetminus \{P\}$, but $\nu(g) = 0$ so $g \in \mathcal{O}_{P,C}$, and g is regular at P as well. □

The t of the corollary is the algebraic substitute for a local coordinate on C.

2. Extending morphisms from curves

A basic result on morphisms from curves to varieties is the following.

Theorem 2.1. *If C is a curve, and $P \in C$ a nonsingular point, and Y a projective variety, then every morphism $C \smallsetminus \{P\} \to Y$ extends uniquely to a morphism $C \to Y$.*

Remark 2.2. We see that the uniqueness is satisfied for Y any variety, and indeed is a consequence of the condition analogous to being Hausdorff which we used to distinguish varieties among prevarieties. Thus, we need only to prove the existence statement for the theorem.

Remark 2.3. The idea of extending morphisms of nonsingular curves as in the theorem plays an important role in algebraic geometry, more or less replacing the use of convergent sequences in metric topology. Even though we don't know what it means for points to be close over an arbitrary field, if $\varphi : C \smallsetminus \{P\} \to Y$ extends as in the theorem we can think of $\varphi(P)$ as representing $\lim_{Q\to P} \varphi(Q)$. In particular, we can define a notion analogous to compactness using existence of such extensions; in this language, the theorem is saying that projective varieties are compact, because limits always exist. We also see that the defining condition for varieties can be thought of as saying that limits are unique, when they exist (this is certainly a necessary condition for being Hausdorff; for prevarieties, it turns out to be sufficient, as well).

Proof. By Remark 2.2, it suffices to prove the existence of the desired extension. Let $\varphi : C \smallsetminus \{P\} \to Y$ be the given morphism. Let $U \ni P$ be an open subset such that there exists a t as in Corollary 1.5, and such that on $U \smallsetminus \{P\}$, we can represent φ by an $(n+1)$-tuple of regular functions $f_0, \dots, f_n \in \mathcal{O}(U \smallsetminus \{P\})$ which do not simultaneously vanish anywhere on $\mathcal{O}(U \smallsetminus \{P\})$.

By Corollary 1.5, we can write each f_i as $t^{e_i} g_i$, where e_i is the valuation of f_i at P (possibly negative), and g_i is regular on U, with $g_i(P) \neq 0$. Choose j with e_j minimal; then since $(f_0, \dots, f_n)$ represents φ on $U \smallsetminus \{P\}$, and t is nonvanishing on this subset, scaling simultaneously by t^{-e_j} we find that $(t^{e_0 - e_j} g_0, \dots, t^{e_n - e_j} g_n)$ also represents φ on the same subset. But $e_i \geqslant e_j$ for all i, so these functions are regular on all of U, and $t^{e_i - e_i} g_i$ is non-zero at P, so setting $\varphi(P) = (t^{e_0 - e_j} g_0(P), \dots, t^{e_n - e_j} g_n(P))$ gives an extension of φ to U.

Finally, being a morphism is a local condition, so if we have extended φ to a morphism on U, since it was already a morphism on $C \smallsetminus \{P\}$, we conclude that we have extended φ to a morphism on all of C. □

Corollary 2.4. *If two nonsingular projective curves are birational, then they are isomorphic.*

Proof. We have an isomorphism of open subsets, but by the theorem each map extends to a morphism on the whole curve. □

3. Quasiprojectivity

We will now prove the following theorem:

Theorem 3.1. *If C is a nonsingular curve, then C is quasiprojective.*

For the proof, we need one key background statement, which we organize into exercises.

Exercise 3.2. Show that if X is a variety, and for some $P, Q \in X$ we have $\mathcal{O}_{P,X} \subseteq \mathcal{O}_{Q,X} \subseteq K(X)$, then $P = Q$. (Hint: this is not true for an arbitrary prevariety. Compare to Exercise I.4.7 of [1])

Exercise 3.3. (a) Show that if $\varphi : X \to Y$ is a morphism of varieties, and $U \subseteq X$ is an open subset such that the composition $U \to Y$ is an isomorphism, then $U = X$.

(b) Show that if $\varphi : X \to Y$ is a morphism of varieties, and $U \subseteq X$ is an open subset such that $\varphi : U \to Y$ is an isomorphism onto an open subset $V \subseteq Y$, then $\varphi^{-1}(V) = U$.

Proof of Theorem 3.1. Let U_i be a cover of C by affine open subsets. Then we have $U_i \subseteq \mathbb{A}^{n_i} \subseteq \mathbb{P}^{n_i}$, so we take Y_i to be the closure of U_i in $\mathbb{P}^{n_i}$. Thus Y_i is projective, and U_i is isomorphic to an open subset of Y_i. By Theorem 2.1, we obtain unique extensions $\varphi_i : C \to Y_i$ for each i (note that each U_i may omit more than one point of C, but we can apply the theorem inductively to extend over each one). These extensions may not be isomorphisms onto their images, because we have little control over what happens when we take the closure of U_i. The trick is to take the product over all i; we then have an induced morphism $\varphi : C \to \prod_i Y_i \subseteq \prod_i \mathbb{P}^{n_i}$. Let $Y \subseteq \prod_i Y_i$ be the closure of the image of C. We will show that C is isomorphic to an open subset of Y. This will prove the theorem, because Y is a closed subset of $\prod_i \mathbb{P}^{n_i}$, which is itself projective via the Segre imbedding (see Exercises 2.14, 3.16 of [1]).

Our first task is to show that φ is a homeomorphism onto an open subset of Y. Now, φ is injective, since given $P, Q \in C$, if $P \in U_i$, we claim $\varphi_i(P) \neq \varphi_i(Q)$. If $Q \in U_i$ as well, this follows from the injectivity of φ_i on U_i, but if $Q \notin U_i$, then $\varphi_i(Q) \notin U_i \subseteq Y_i$ by Exercise 3.3 (b), while $\varphi_i(P) \in U_i$, proving the claim, and thus injectivity. It then follows that φ is a homeomorphism onto its image, since φ maps finite sets to finite sets and thus closed subsets of C to closed subsets of $\varphi(C)$. We next observe that φ is dominant onto Y by definition, so Y is irreducible, and we have $K(Y_i) \hookrightarrow K(Y) \hookrightarrow K(C)$. But $C \to Y_i$ is birational, to $K(Y_i) = K(C)$, and we conclude that $K(Y) = K(C)$, so $\varphi : C \to Y$ is birational. In particular, we conclude that $K(Y)$ has transcendence degree 1, so Y is a curve, and also that $\varphi(C)$ contains an open subset of Y. But since Y is a curve, any subset containing a nonempty open subset is open, so $\varphi(C)$ is open, and we have proved that φ induces a homeomorphism from C onto an open subset of Y.

We now want to see that φ is an isomorphism of C onto its image. It suffices to show that the induced maps on local rings are isomorphisms at every point of C, so let $P \in C$, and consider the induced map $\mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,C}$. This is injective since $C \to Y$ is dominant. Choose i with $P \in U_i$. Then the map $\varphi_i : U_i \to Y_i$ is an isomorphism onto its image, so the induced map $\mathcal{O}_{\varphi_i(P),Y_i} \to \mathcal{O}_{P,U_i}$ is an isomorphism. But we can factor φ_i as $U_i \hookrightarrow C \to Y \to Y_i$, where the last morphism is projection onto the ith factor from the product. This means that the map $\mathcal{O}_{\varphi_i(P),Y_i} \to \mathcal{O}_{P,U_i}$ factors through $\mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,C} = \mathcal{O}_{P,U_i}$, so we conclude that the latter must be surjective, and hence as isomorphism, as desired. ☐

Remark 3.4. In fact, the theorem holds without the nonsingularity hypothesis, but the proof is a bit more involved. One approach is to show that even on a singular curve, one has an affine open cover such that for every open subset, the omitted points are nonsingular. Given that, the above argument goes through unmodified.

Now that we know that every nonsingular curve is quasiprojective, we can consider the question of projectivity. Obviously, not every nonsingular curve is projective. But we now see that every nonsingular curve can be "compactified" as an open subvariety of a projective curve by imbedding in projective space and taking the closure. But this closure will not in general be nonsingular. So we can ask whether every nonsingular curve can be realized as an open subvariety of a *nonsingular* projective curve. For the moment, although it is clear that every curve is birational to a nonsingular curve, and also to a projective curve, it is not even clear that every curve is birational to a nonsingular projective curve. We will prove the stronger assertion, but only after a discussion of normalization.

4. Normality and normalization

We make a brief detour to discuss the notion of normality. Most of the proofs, while not necessarily difficult, are purely algebraic, and we omit them.

Definition 4.1. A variety X is **normal** if it covered by affine open subvarieties U_i such that each $A(U_i)$ is integrally closed in $K(X)$.

Recall that given an inclusion of integral domains $R \subseteq S$, an element $s \in S$ is *integral* over R if it is a root of a monic polynomial with coefficients in R. We say R is *integrally closed* in S if every element of S which is integral over R is in fact an element of R. The integers are integrally closed in the rational numbers, by Gauss's lemma, motivating the terminology.

Normality is a somewhat subtle condition, but it does have a fairly direct relationship to nonsingularity. Specifically, we have:

Proposition 4.2. *A nonsingular variety is normal. The singular locus of a normal variety has codimension at least* 2.

The proof of this is fairly standard algebra, and we omit it. Since any non-empty closed subset of a curve has codimension at most 1, we conclude:

Corollary 4.3. *A normal curve is nonsingular.*

Another basic algebra statement is:

Proposition 4.4. *X is normal if and only if every affine open subset U has $A(U)$ integrally closed in $K(X)$, if and only if $\mathcal{O}_{P,X}$ is integrally closed in $K(X)$ for all points $P \in X$.*

If an integral domain R is not integrally closed in its field of fractions, we can take the integral closure, which is the set of all elements of the field which are integral over R. More generally, if $R \subseteq S$ is not integrally closed in S, we can take its integral closure in S. It is a basic algebra fact that the integral closure is again a subring, and it is integrally closed in the field of fractions. The proof is related to a fact we have already used earlier, that f is integral over R if and only if the ring $R[f]$ is a finitely-generated R-module.

We are now ready to define the normalization. It will be useful to give a slightly more general form than is immediately necessary.

Definition 4.5. If X is a variety, and L is a finite field extension of $K(X)$ (in particular, algebraic over $K(X)$) the **normalization** $\tilde{X}_L \to X$ of X in L is the variety constructed by taking an affine open cover U_i of X, letting $\tilde{U}_i$ be the affine variety determined by the integral closure of $A(U_i)$ in L, and gluing according to injections $A(\tilde{U}_i) \hookrightarrow L$.

The morphism $\tilde{X}_L \to X$ is the morphism induced by the inclusions $A(U_i) \hookrightarrow A(\tilde{U}_i)$.

In particular, the **normalization** $\tilde{X} \to X$ of X is the normalization of X in $K(X)$.

To make sense of the gluing statement, we have the following basic algebra result:

Exercise 4.6. If R is an integral domain with fraction field K, and L an algebraic extension of K, then the integral closure of R in L has fraction field L.

We then have that each $A(\tilde{U}_i)$ has fraction field L, so the injections into L induce isomorphisms of fraction fields, and hence birational maps $\tilde{U}_i \dashrightarrow \tilde{U}_j$ for all i, j. These birational maps in turn correspond to isomorphisms between open subsets, which we use to define the gluing.

Theorem 4.7. *The normalization $\tilde{X}_L$ of X in L is a normal variety, and independent of the choice of U_i. The morphism $\tilde{X}_L \to X$ is surjective, and the induced map on function fields is $K(X) \hookrightarrow K(\tilde{X}_L) = L$.*

If $U \subseteq X$ is open, and $\tilde{X}_L \to X$ and $\tilde{U}_L \to U$ the respective normalizations, then $\tilde{U}_L$ is naturally identified with the preimage of U in $\tilde{X}_L$.

Note that in particular, the normalization of X yields a birational morphism.

The main point of the proof is the (quite non-trivial) algebra theorem that for R a finitely generated k-algebra, the integral closure in a finite field extension is finitely generated over R.

Remark 4.8. We start to see the utility of having a notion of abstract variety: while it is true that the normalization of a projective variety is projective, the proof isn't trivial, and something of a distraction from the basic idea, that we are simply gluing together integral closures.

Example 4.9. Consider the cuspidal curve $C \subsetneq \mathbb{A}^2$ given by $y^2 = x^3$. This is a singular curve, so not normal. We have studied the morphism $\mathbb{A}^1 \to C$ given by $t \mapsto (t^2, t^3)$, corresponding to the injective homomorphism $k[x, y]/(y^2 - x^3) \to k[t]$ sending x to t^2 and y to t^3. This homomorphism induces an isomorphism on fraction fields (this follows from the observation t is the image of $\frac{y}{x}$). We see that t is integral over $A(C)$, since it satisfies $z^2 - x$ (and also $z^3 - y$). But $k[t]$ is integrally closed in its fraction field (one may check this directly, or invoke that nonsingularity implies normality), so we conclude that the morphism $\mathbb{A}^1 \to C$ is in fact the normalization of C.

We see that normalization is universal for (dominant) morphisms from normal varieties:

Proposition 4.10. *Suppose $\varphi : Y \to X$ is a dominant morphism, with Y normal. Then φ factors through the normalization map $\tilde{X} \to X$.*

Proof. Cover X by affine open subsets U_i, and let V_j be a affine open cover of Y such that each V_j is contained in some $\varphi^{-1}(U_i)$. Let $\tilde{U}_i$ be the preimages of U_i in $\tilde{X}$. Fix i, j with $V_j \subseteq \varphi^{-1}(U_i)$. Note that the dominance of φ implies that $A(U_i) \subseteq A(V_j)$. Also $A(U_i) \subseteq A(\tilde{U}_i)$ by definition. We can consider both these inclusions to hold inside $K(Y)$.

We claim that in fact $A(\tilde{U}_i) \subseteq A(V_j)$ in $K(Y)$. This follows immediately from the definitions; $A(\tilde{U}_i)$ is the set of elements of $K(X)$ which are integral over $A(U_i)$, and by the hypothesis that V_i is normal implies that $A(V_j)$ contains all elements of $K(Y)$ which are integral over $A(V_j)$. But because $K(X) \subseteq K(Y)$ and $A(U_i) \subseteq A(V_j)$, any element of $K(X)$ integral over $A(U_i)$ is in particular an element of $K(Y)$ integral over $A(V_j)$, so must lie in $A(V_j)$. We conclude that $A(\tilde{U}_i) \subseteq A(V_j)$, so the morphisms $V_j \to U_i$ factor through $\tilde{U}_i \to U_i$. Thus, for each V_j we get that $V_j \to X$ factors through $\tilde{X}$.

But given j, j' the resulting morphisms $V_j \to \tilde{X}$ and $V_{j'} \to \tilde{X}$ both induce the same inclusion $K(X) \to K(Y)$ of function fields by construction, so they define the same rational maps, and agree on $V_j \cap V_{j'}$. We can thus glue them all together to obtain the desired morphism $Y \to \tilde{X}$. □

5. Projective curves

The main utility of normalization for us is that it provides a method of desingularizing curves, and we see that it preserves projectivity.

Theorem 5.1. *The normalization of a projective curve is a nonsingular projective curve.*

Proof. Let C be the projective curve, and $\tilde{C}$ its normalization. Nonsingularity of $\tilde{C}$ is immediate from Corollary 4.3, while we know that $\tilde{C}$ is quasiprojective from Theorem 3.1. We claim that if we have $\tilde{C} \subseteq \mathbb{P}^n$, it must be closed, so that $\tilde{C}$ is projective, as desired. Let Y be the closure of $\tilde{C}$ in $\mathbb{P}^n$. Given $P \in Y$, let U be an affine neighborhood of P in Y, and $\tilde{U}$ its normalization. Then since $\tilde{U}$ is a nonsingular curve and C is projective, by Theorem 2.1 the birational map $\tilde{U} \dashrightarrow C$ induced by

$$\tilde{U} \to U \hookrightarrow Y \dashrightarrow \tilde{C} \to C$$

extends to a morphism $\tilde{U} \to C$.

Because $\tilde{U}$ is normal, it follows from Proposition 4.10 that this morphism factors through $\tilde{C} \to C$. By construction, the induced morphism $\tilde{U} \to \tilde{C} \hookrightarrow Y$ agrees with the composed morphism $\tilde{U} \to U \hookrightarrow Y$ on an open subset, and hence on all of $\tilde{U}$. But by surjectivity of normalization, there is some $\tilde{P} \in \tilde{U}$ mapping to $P \in U$, and if we let Q is its image in $\tilde{C}$, we conclude $Q = P$ in Y, so $P \in \tilde{C}$. Since P was arbitrary in Y, we conclude that $\tilde{C} = Y$, and $\tilde{C}$ is projective, as desired. □

Corollary 5.2. *Let C be a nonsingular curve. Then there is a nonsingular projective curve $\bar{C}$ (necessarily unique) such that C is isomorphic to an open subvariety of $\bar{C}$.*

Proof. By Theorem 3.1, we can realize C as a quasiprojective curve. Let Y be its closure in projective space. By Theorem 5.1, if $\bar{C}$ is the normalization of Y, it is a projective nonsingular curve. Finally, since C is a nonsingular open subset of Y (and using the assertion on restriction to open subsets in Theorem 4.7), the normalization map $\bar{C} \to Y$ is an isomorphism on C, so we have that C is isomorphic to an open subvariety of $\bar{C}$, as desired. □

Corollary 5.3. *Every curve is birational to a unique nonsingular projective curve.*

Proof. The uniqueness is Corollary 2.4. Given any curve C, we know it has a non-empty open subset U of nonsingular points, so applying Theorem 5.2 to U, we can imbed it into a nonsingular projective curve, which is then birational to C. □

Another consequence of the normalization construction is the following:

Corollary 5.4. *If C is a projective curve, D is any curve, and $\varphi : C \to D$ a non-constant morphism, then φ is surjective.*

Proof. Since φ is non-constant, it must be dominant: indeed, if $\varphi(C)$ is contained in a proper closed subset of D, it must be a single point, since C (and hence $\varphi(C)$) is connected. It thus induces an injection $K(D) \hookrightarrow K(C)$, and since both funtion fields have transcendence degree 1, we conclude $K(C)$ is algebraic over $K(D)$. Since they are both finitely generated over k, $K(C)$ is finitely generated over $K(D)$, so is a finite extension. We can thus let $\tilde{D}_C$ be the normalization of D in $K(C)$. Now, $\tilde{D}_C$ is birational to C, so we have a rational map $\psi : \tilde{D}_C \dashrightarrow C$ such that $\varphi \circ \psi$ is equal to the normalization map (considered as a rational map). But because $\tilde{D}_C$ is nonsingular and C is projective, ψ extends to a morphism which must satisfy the same relation that $\varphi \circ \psi$ is equal to the normalization. But the normalization map is surjective, so we conclude that φ is likewise surjective. □

Corollary 5.5. *If $\varphi : C \to D$ is a morphism of curves, then $\varphi(C)$ is an open subset of D.*

Proof. Since D has the cofinite topology, it is enough to show that $\varphi(C)$ contains an open subset of D. We may therefore restrict to the sets of nonsingular points of both C and D, so we reduce to the case that C and D are nonsingular. Now let $\bar{C}$ and $\bar{D}$ be the nonsingular projective compactification of C and D provided by Corollary 5.2. Then φ extends to a morphism $\bar{C} \to \bar{D}$, which we know is surjective by Corollary 5.4. But $\bar{C} \smallsetminus C$ consists of a finite set of points, so $\varphi(C)$ contains all but a finite set of points of $\bar{D}$, and in particular of D, so we conclude it is an open subset of D, as desired. □

Remark 5.6. Both Corollary 5.4 and Corollary 5.5 generalize to higher-dimensional varieties as follows: the image of a morphism from a projective variety to a variety is always closed, and the image of any dominant morphism of varieties contains an open subset of the target. The latter (is a slightly strengthened form) is known as Chevalley's theorem. We will prove both statements shortly.

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.