RECOVERING GEOMETRY FROM CATEGORIES

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The fundamental organizing result on affine algebraic varieties over an algebraically closed field k is surely that the category they form is equivalent to the (opposite) category of finitely generated integral domains over k. This single statement encapsulates the facts that two affine varieties are isomorphic if and only if their coordinate rings are isomorphic, and more generally, that every morphism of affine varieties come from a homomorphism (in the other direction) of their coordinate rings.

However, this abstract categorical statement strips away the rich geometry enjoyed by the varieties themselves. At first, it may also seem as though abstract statements about the category cannot possibly capture the geometric content of its objects. And surely, the geometric definitions and concepts remain an important and useful foundation in the study of varieties, even as the translation to algebra provides crucial tools. Nonetheless, it is the goal of this note to demonstrate that a surprising amount of the geometry of varieties can be recovered from the abstract "dots and arrows" information encoded by the category. This underlines the power of the fundamental categorical equivalence.

Let k be an algebraically closed field. We denote by $\mathbf{Var}/k$ the category of affine algebraic varieties over k. Then the statement we will prove is the following.

Proposition 1. *Let X be an object of $\mathbf{Var}/k$; that is, an affine variety. Then the set of points of X, as well as the Zariski topology on it, can be recovered completely from the data of the category $\mathbf{Var}/k$.*

We assume the reader is familiar with the definition of a category, with final and initial objects, and monomorphisms and epimorphisms. The affine variety consisting of a single point will play an important role.

Definition 2. The **point**, denoted $*$, is the isomorphism class of a single point.

Note that any two affine varieties which are a point are isomorphic, in fact uniquely.

We first see:

Lemma 3. *The point is the universal final object of $\mathbf{Var}/k$.*

Proof. This is tautological if one allows $\mathbb{A}^0$ as an object of $\mathbf{Var}/k$, but even if not, it is clear that for any representative of $*$ as a subvariety of $\mathbb{A}^n$, every affine variety X has a unique function $X \to *$, which is a morphism. □

Corollary 4. *The set underlying an affine variety X may be recovered from $\mathbf{Var}/k$.*

Proof. Indeed, we know from Lemma 3 that $*$ is recognizable from the structure of $\mathbf{Var}/k$. If we choose any representative of $*$, we see that the set of morphisms $* \to X$ are in bijection with the set of points of X, since for any $P \in X$, there is clearly a unique function $* \to X$ with image P, and this can be described by polynomials as a constant function, so is a morphism. □

We next address the question of recovering the topology on X. We recall a relevant definition from category theory.

Definition 5. A morphism $h : A \to B$ in a category is an **extremal epimorphism** if it is an epimorphism such that if $h = f \circ g$, and f is a monomorphism, then f is an isomorphism. Similarly, h is an **extremal monomorphism** if it is a monomorphism such that if $h = f \circ g$, and g is an epimorphism, then g is an isomorphism.

We will now start working with rings as well, so let **Alg**$/k$ denote the category of finitely generated integral domains over k.

Lemma 6. *In the category* **Alg**$/k$*, a morphism $h : A \to B$ is a monomorphism if and only if it is injective. It is an extremal epimorphism if and only if it is surjective.*

Proof. Since a ring homomorphism is determined by the map on underlying sets, it is clear that an injective map is a monomorphism, and a surjective map is an epimorphism, and indeed an extremal epimorphism. Conversely, suppose $h : A \to B$ has kernel I, and $a \in I$ is non-zero. Then consider the homomorphisms $g_1, g_2 : k[t] \to A$ determined by sending t to 0 or to a. Then $h \circ g_1 = h \circ g_2$, so h is not a monomorphism. This establishes the first assertion. On the other hand, suppose $h : A \to B$ is not surjective. Then h factors as $h : A \to h(A) \to B$, with the latter homomorphism injective, hence a monomorphism. Thus h is not an extremal epimorphism. □

Remark 7. The proof shows that for h to be surjective is in fact equivalent to the condition that if $h = f \circ g$, and f is a monomorphism, then f is an isomorphism (that is, we don't need to separately require that h is an epimorphism).

We need one more definition.

Definition 8. We say that a morphism $Y \to X$ of affine varieties is a **closed imbedding** if it factors $Y \to Z \to X$ where $Y \to Z$ is an isomorphism, and Z is a closed subvariety of X.

This is now enough for us to prove the main result.

Proof of Proposition 1. Our first observation is that if X is an affine variety, and Y is another affine variety, with $f : Y \to X$ a morphism, then we can recover $f(Y) \subseteq X$ as the set of morphisms $* \to X$ which factor through Y. Next, it follows immediately from the definition of the Zariski topology and our correspondence theorems that $f : Y \to X$ is a closed imbedding if and only if $f^\sharp : A(X) \to A(Y)$ is surjective. By Lemma 6, this is equivalent to $f^\sharp$ being an extremal epimorphism. By our categorical equivalence, we see that f is a closed imbedding if and only if it is an extremal monomorphism. But this is a purely categorical condition. We can thus reconstruct the Zariski topology on X by setting the closed subsets to be finite unions of images $f(Y) \subseteq X$, where $f : Y \to X$ is any extremal monomorphism in **Var**$/k$. □