MATH 248A TAKEHOME EXAM

This exam is due by 5:00 PM on Wednesday, December 9. You are free to consult your notes and Hartshorne, but are not allowed to discuss the exam with one another. Feel free to email me with questions.

Problem 1. Suppose that $f_{d-1}, f_d \in k[x_1, \ldots, x_n]$ are homogeneous polynomials of degree $d-1$ and d, respectively. Show that if $X = Z(f_{d-1} + f_d) \subseteq \mathbb{A}^n$ is irreducible, it is birational to $\mathbb{A}^{n-1}$. Hint: first look at a projection from a carefully chosen point.

Problem 2. Assume that $\operatorname{char} k \neq 2$, and let $X \subseteq \mathbb{A}^2$ be the affine plane curve defined by

$$y^2 = g(x)$$

for some polynomial g.

(a) Give necessary and sufficient conditions in terms of $g(x)$ for X to be nonsingular.

For the remainder of the problem, assume X is nonsingular.

(b) Consider the morphism $X \to \mathbb{A}^1$ defined by projection to the x-axis. What is the degree of this morphism? What are the ramification points, and their ramification indices?

(c) Let Y be the closure of X in $\mathbb{P}^2$. Give necessary and sufficient conditions in terms of $g(x)$ for Y to be nonsingular.

(d) Let $\tilde{Y}$ be the normalization of Y. Consider the morphism $\tilde{Y} \to \mathbb{P}^1$ obtained by extending the morphism $X \to \mathbb{A}^1$ of (b). Use the Riemann-Hurwitz formula

$$2d + 2g - 2 = \sum_{P \in \tilde{Y}} (e_P - 1)$$

(where g is the genus of $\tilde{Y}$) to find g and to determine the ramification indices of the point(s) over ∞.

Problem 3. Let X be a projective nonsingular curve. Show that there is no nondegenerate morphism from X to $\mathbb{P}^7$ of degree 5.

Problem 4. Let X be a projective nonsingular curve, and let D be a divisor of degree 1. Suppose that $\ell(D) = 2$. Show that X is isomorphic to $\mathbb{P}^1$.