HOMEWORK 1
SELECTED SOLUTIONS

1.1 a) Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.
b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

Proof: a) We first observe that because $Y = Z(y - x^2)$, in order to prove that $I(Y) = (y - x^2)$, it is enough to check that $(y-x^2)$ is radical. We will show that $k[x, y]/(y-x^2) \cong k[x]$, from which it follows that $(y - x^2)$ is prime, hence radical. We then conclude that $I(Y) = (y - x^2)$, so $A(Y) \cong k[x]$, as desired. Consider the map $\phi : k[x, y]/(y - x^2) \to k[x]$ which sends $f(x, y)$ to $f(x, x^2)$, and the map $\psi : k[x] \to k[x, y]/(y - x^2)$ which sends $g(x)$ to itself. It is clear that both maps are well-defined homomorphisms, and that $\phi(\psi(g)) = g$ for any $g \in k[x]$. For any $f(x, y) \in k[x, y]/(y - x^2)$, we have $\psi(\phi(f(x, y))) = \psi(f(x, x^2)) = f(x, x^2)$, which is equal to $f(x, y)$ in $k[x, y]/(y - x^2)$ since $k[x, y] \to k[x, y]/(y - x^2)$ is a homomorphism sending y and x^2 to the same element. Therefore, ϕ and ψ are inverse to one another, and we conclude $k[x] \cong k[x, y]/(y - x^2) \cong A(Y)$.
Remark: there are several alternatives to this argument that would be reasonable. To prove $I(Y) = (y-x^2)$, one could argue directly that $y-x^2$ is irreducible, and hence generates a prime ideal of $k[x, y]$. One could argue directly that either of the maps is a bijective ring homomorphism. You should not expect full credit on a problem, however elementary it may seem, for a one-line answer amounting to a restatement of what you are asked to prove. If you feel you are above the sort of argument above, you could quite reasonably instead prove a lemma as follows, which you could then cite in subsequent problems. Moral: I want to see evidence that you have thought through which parts really are immediate from the definitions, and which are conceptually clear but may require some sort of argument.

Lemma 1. *Suppose R is a ring and $r \in R$. Then $R[x]/(x - r) \cong R$.*

Proof. Consider the homomorphism $\psi : R \to R[x]/(x - r)$ which sends $s \in R$ to itself. This is surjective, because $f(x)$ is equivalent to $f(r)$ modulo $x - r$. Moreover, it is injective because any non-zero multiple of $x - r$ in $R[x]$ necessarily has a leading term with a strictly positive power of x. □

b) As in (a), we have $Z = Z(xy - 1)$, and it follows that $I(Z) = (xy - 1)$ and $A(Z) = k[x, y]/(xy - 1)$ if we know that $(xy - 1)$ is radical. But if $xy - 1$ is a product of non-constant polynomials, by considering highest degrees with respect to x and y we see that we must have a linear polynomial in x times a linear polynomial in y, and such a product cannot give $xy - 1$. Since $k[x, y]$ is a unique factorization domain, it follows that $(xy - 1)$ is prime, so $k[x, y]/(xy - 1)$ is equal to $A(Z)$.
Now, let $\phi : A(Z) \to k[z]$ be a homomorphism. If e is a unit in $A(Z)$, its image is necessarily a unit in $k[z]$, hence a constant. But $A(Z)$ is generated as a ring by k^*, x and y, which are all units, so we conclude $\phi(A(Z)) \subseteq k \subseteq k[z]$. Thus, ϕ could not be an isomorphism.

1.8 Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r-1$.

Proof: Let f be the equation defining H. Since $Y \not\subseteq H$, we have $f \notin I(Y)$, and $f \neq 0$ in $A(Y)$. Note that $A(Y)$ is an integral domain. Thus, f is not a zero divisor in $A(Y)$. Now, if f is a unit in $A(Y)$, we conclude that $Z(f) \cap Y = H \cap Y = \emptyset$, so the statement is vacuously true. We thus may suppose that f is also not a unit, so Krull's principal ideal theorem tells us that every minimal prime ideal of $A(Y)$ containing f has height 1. If Z is an irreducible component of $Y \cap H$, then $I(Z)$ is a minimal prime ideal containing $I(Y \cap H) = (I(Y), f)$, so its image in $A(Y)$ is a minimal prime ideal containing f, and therefore has height 1. However, we also have that $\operatorname{codim}_Y Z = \operatorname{height}_{A(Y)} I(Z)$ and $\dim Z + \operatorname{codim}_Y Z = \dim Y$. Thus, we conclude that $\dim Z = r-1$.

1.12 Given an example of an irreducible polynomial $f \in \mathbb{R}[x,y]$, whose zero set $Z(f)$ in $\mathbb{A}^2_{\mathbb{R}}$ is not irreducible.

Solution: Let $f = y^2 + (x^2-1)^2$. The zero set $Z(f) = \{(0,1),(0,-1)\}$ is not irreducible, as any point is an algebraic set.

$f = (y + i(x^2-1))(y - i(x^2-1))$. As in earlier problems (see also the optional lemma), we see that $g = y \pm i(x^2-1) \in \mathbb{C}[x,y]$ is irreducible. By unique factorization in $\mathbb{C}[x,y]$, we have that f cannot factor in $\mathbb{R}[x,y]$.

Note You may use prior exercises that we have not done, as long as you cite them. For instance, you could cite 1.6 in solving 1.10(a).