HOMEWORK 2
SELECTED SOLUTIONS

3.2 A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.

a) For example, let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism.

b) For another example, let the characteristic of the base field k be $p > 0$, and define a map $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ by $t \mapsto t^p$. Show that φ is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism.*

Proof: a) The image of φ is clearly contained in $Y := Z(y^2 - x^3)$. We claim it is bijective onto this curve: it is injective since if $x = t^2 \neq 0$, then t is determined up to sign, and changing the sign will change $y = t^3$. If $x = 0$, then $t = 0$ necessarily. Moreover, it is surjective since if we have (x, y) with $y^2 = x^3$, and we let t be a square root of x, then $t^6 = x^3 = y^2$, so (x, y) is either the image of t or of $-t$. Since Y is the continuous image of an irreducible space, it is irreducible, so it is indeed a variety. Since φ is defined by polynomials, it determines a morphism $\mathbb{A}^1 \to Y$, which is then necessarily continuous. Moreover, we see it is a closed map, since it is surjective, and maps finite sets to finite sets. Thus, φ is a bijective bicontinuous morphism.

To see that φ is not an isomorphism, we see that it does not induce an isomorphism on coordinate rings. But $A(Y)$ is a quotient of $k[x, y]$, and the induced map $k[x, y] \to k[t]$ is given by $x \mapsto t^2, y \mapsto t^3$. Thus the linear term of anything in the image always vanishes, so the map is not surjective.

Remark: We could show directly that $y^2 - x^3$ is irreducible, so defines a variety (and thus Y has coordinate ring $k[x, y]/(y^2 - x^3)$). Remarkably, this is not necessary to know for the exercise. Also, note that it is easier to see that φ doesn't induce an isomorphism on coordinate rings than to prove that the coordinate rings are not isomorphic (although they're not, and it's not terribly difficult to prove that).

b) This is evidently a morphism, since it is given by a polynomial. To see it is injective, we use that if $t^p = a$, and b is a pth root of a, then $0 = t^p - a = t^p - b^p = (t - b)^p$ in positive characteristic, and since $k[t]$ is still a UFD, we conclude b is the only pth root of a. Moreover, it is surjective, since k is algebraically closed, so for any a, there exists some pth root b. It is automatically continuous, and it is bicontinuous since it is closed by the same argument as for a). But it is not an isomorphism, since the induced map on coordinate rings is again not surjective, as the image only includes exponents which are multiples of p.

3.15 Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties.

a) Show that $X \times Y \subseteq \mathbb{A}^{n+m}$ with its induced topology is irreducible.

b) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

c) Show that $X \times Y$ is a product in the category of varieties.

d) Show that $\dim X \times Y = \dim X + \dim Y$.

Proof: a) Suppose $X \times Y = Z_1 \cup Z_2$ for Z_1, Z_2 closed in $X \times Y$. Let $X_i = \{x \in X : x \times Y \subseteq Z_i\}$. We claim the X_i are closed subsets. Suppose $x \notin X_i$. Then there exists $y \in Y$ with $(x, y) \notin Z_i$, by construction. Now, $X \times \{y\}$ is clearly closed in $X \times Y$, so $Z_i \cap (X \times \{y\})$ is closed, and doesn't contain (x, y). Moreover, the image of $Z_i \cap (X \times \{y\})$ in X contains X_i, and because $X \times \{y\}$ is homeomorphic to X, the image in X is closed. But by construction, x is not in the image of $Z_i \cap (X \times \{y\})$, so we conclude it is not in the closure of X_i. We conclude that X_i is closed.

Our next claim is that $X = X_1 \cup X_2$. Given $x \in X$, $(\{x\} \times Y) \cap Z_i$ is, as above, a closed subset of $X \times Y$ with image in Y a closed subset of Y. Moreover, the union of the images for $i = 1, 2$ is all of Y, since by hypothesis $Z_1 \cup Z_2 = X \times Y$. By irreducibility of Y, we conclude one image must be all of Y, meaning that $\{x\} \times Y \subseteq Z_i$ for $i = 1$ or 2. Then $x \in X_i$, and the claim follows. Finally, by irreducibility of X we conclude that $X = X_i$ for $i = 1$ or 2, and then that $Z_i = X \times Y$, as desired.

Remark: The complement of X_i is the image of the complement of Z_i, so proving X_i is closed amounts to proving the map $X \times Y \to X$ is an open map. However, it is not a closed map, as we have seen with the image of the hyperbola under $\mathbb{A}^2 \to \mathbb{A}^1$.

b) For clarity of notation, we use x_i for coordinates on $\mathbb{A}^n$, and y_j for coordinates on $\mathbb{A}^m$, and then consider the coordinates on $\mathbb{A}^{n+m}$ to be $x_1, \ldots, x_n, y_1, \ldots, y_m$. We define a homomorphism $\varphi : A(X) \otimes_k A(Y) \to A(X \times Y)$ as follows: given a tensor $\bar{f} \otimes \bar{g}$, we choose lifts f, g of $\bar{f}, \bar{g}$ to the respective polynomial rings, and set $\varphi(\bar{f} \otimes \bar{g})$ to be the image in $A(X \times Y)$ of $f(x_1, \ldots, x_n) \cdot g(y_1, \ldots, y_m)$. We define φ by extending linearly. This definition is independent of the choices of f, g because if for instance $f - h \in I(X)$, then $(f - h)(x_1, \ldots, x_n) \in I(X \times Y)$ when considered as a function of $n + m$ variables. Moreover, φ is well-defined on the tensor product because it is k-linear on both sides, and by the usual distributive rules for multiplication. It is also clear that it preserves addition and multiplication, and sends 1 to 1, so it is a homomorphism.

It remains to show that φ is bijective. Surjectivity is clear, since any element of $A(X, Y)$ can be lifted to a polynomial, which is of the form

$$\sum_{\mathbf{i},\mathbf{j}} c_{\mathbf{i},\mathbf{j}} \mathbf{x}^{\mathbf{i}} \cdot \mathbf{y}^{\mathbf{j}},$$

and therefore equal to

$$\varphi(\sum_{\mathbf{i},\mathbf{j}} c_{\mathbf{i},\mathbf{j}} \mathbf{x}^{\mathbf{i}} \otimes \mathbf{y}^{\mathbf{j}})$$

(here we use bold to denote tuples of indices and products of tuples of variables).

Finally, an arbitrary element of $A(X) \otimes A(Y)$ is of the form $\sum_i \bar{f}_i \otimes \bar{g}_i$. Using the tensor product relations, we see that either $\sum_i \bar{f}_i \otimes \bar{g}_i = 0$, or we may assume that any two $\bar{f}_i$ are linearly independent over k, and that no $\bar{g}_i$ is equal to 0. Indeed, if any $\bar{g}_i = 0$, we may always remove that term, by the definition of the tensor product. If some $\bar{f}_i$ can be expressed as a linear combination of the others, we can break up the tensor product and redistribute, and we eliminate that $\bar{f}_i$ at the cost of modifying the $\bar{g}_i$. Iterating both steps, either we end up with the empty sum, in which case the tensor was 0 to start with, or we have obtained the desired form. Now suppose that the image $\sum_i \bar{f}_i \bar{g}_i$ is 0 in $A(X \times Y)$. Since the $\bar{g}_i$ are not 0, there is some point $P \in Y$

for which $\bar{g}_1(P) \neq 0$. Then evaluating at P, we find

$$\sum_i \bar{f}_i \bar{g}_i(P) = 0,$$

which contradicts linear independence.

Remark: The construction of φ can be shortened considerably using the universal property of tensor products, and this would be perfectly acceptable.

c) The projection maps are morphisms since they are defined by polynomials (indeed, linear monomials). I should have said it is enough to treat the case that Z is affine, since we had not yet defined morphisms of varieties more generally. It is then clear that a morphism $Z \to X \times Y$ yields morphisms $Z \to X$ and $Z \to Y$, by composing with the projections. Conversely, a pair of morphisms $Z \to X, Z \to Y$ is given by tuples of polynomials $f_1(t_1, \ldots, t_\ell), \ldots, f_n(t_1, \ldots, t_\ell)$ and $g_1(t_1, \ldots, t_\ell), \ldots, g_m(t_1, \ldots, t_\ell)$, so simply by concatenating the tuples we obtain a morphism $Z \to \mathbb{A}^{n+m}$, which visibly factors through $X \times Y$, and has the correct compositions with the projections. Moreover, having the correct compositions with the projections clearly defines the morphism $Z \to X \times Y$ uniquely if it exists, so we are done.

Remark: One could also argue using b) and the universal property of tensor products. This argument goes through if Z is an arbitrary variety, using the more general result in that setting given by Proposition 3.5.

d) Suppose we have chains $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_d$ and $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_e$ of irreducible closed subsets of X and Y respectively. Then we obtain the chain

$$X_0 \times Y_0 \subsetneq X_1 \times Y_0 \subsetneq \cdots \subsetneq X_d \times Y_0 \subsetneq X_d \times Y_1 \subsetneq \cdots \subsetneq X_d \times Y_e$$

in $X \times Y$, and each element of the chain is irreducible by a). It follows that $\dim X \times Y \geq \dim X + \dim Y$.

On the other hand, suppose we have chains $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_d$ and $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_e$ of irreducible closed subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively, with $X \subseteq X_0$ and $Y \subseteq Y_0$. Then we obtain the chain

$$X_0 \times Y_0 \subsetneq X_1 \times Y_0 \subsetneq \cdots \subsetneq X_d \times Y_0 \subsetneq X_d \times Y_1 \subsetneq \cdots \subsetneq X_d \times Y_e$$

in $\mathbb{A}^{n+m}$, with $X \times Y \subseteq X_0 \times Y_0$, so we conclude that $\operatorname{codim}_{\mathbb{A}^{n+m}} X \times Y \geq \operatorname{codim}_{\mathbb{A}^n} X + \operatorname{codim}_{\mathbb{A}^m} Y$. We thus have

$$\begin{aligned} \dim X \times Y &= \dim \mathbb{A}^{n+m} - \operatorname{codim}_{\mathbb{A}^{n+m}} X \times Y \\ &\leq n + m - \operatorname{codim}_{\mathbb{A}^n} X - \operatorname{codim}_{\mathbb{A}^m} Y \\ &= \dim X + \dim Y, \end{aligned}$$

yielding the desired identity.

Remark: It is also possible to argue algebraically using transcendence degrees of the corresponding function fields.