HOMEWORK 3
SELECTED SOLUTIONS

2.9 If $Y \subseteq \mathbb{A}^n$ is an affine variety, we identify $\mathbb{A}^n$ with an open set $U_0 \subseteq \mathbb{P}^n$ by the homeomorphism φ_0. Then we can speak of $\overline{Y}$, the closure of Y in $\mathbb{P}^n$, which is called the *projective closure* of Y.

a) Show that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$, using the notation of the proof of (2.2).

b) Let $Y = \{(t, t^2, t^3) \mid t \in k\} \subseteq \mathbb{A}^3$ be the twisted cubic. Its projective closure $\overline{Y} \subseteq \mathbb{P}^3$ is called the *twisted cubic curve* in $\mathbb{P}^3$. Find generators for $I(Y)$ and $I(\overline{Y})$, and use this example to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do not necessarily generate $I(\overline{Y})$.

Solution: a) First observe that since $\varphi_0^{-1}(Y)$ is closed in U_0, we have $\varphi_0^{-1}(Y) = \overline{Y} \cap U_0$, so $Y = \varphi_0(\overline{Y} \cap U_0)$. In exercise 2.6, we showed that $g(y_1, \ldots, y_n) \in I(Y)$ if and only if $\beta(g) \in I(\overline{Y})$, and it follows immediately that the ideal generated by $\beta(I(Y))$ is contained in $I(\overline{Y})$. Conversely, let $f(x_0, \ldots, x_n)$ be a homogeneous polynomial in $I(\overline{Y})$. Suppose $f = x_0^d g$ for some $d \geq 0$, and such that x_0 is not a factor of g. Then since g vanishes on $\overline{Y} \cap U_0$, which is a nonempty open subset of $\overline{Y}$, and since $Z(g)$ is closed, we have that g vanishes on $\overline{Y}$. Thus, g is in $I(\overline{Y})$. Now let $h = g(1, y_1, \ldots, y_n)$. One verifies that $g = \beta(h)$. Therefore, h is in $I(Y)$. So $f = x_0^d \beta(h)$ is in the ideal generated by $\beta(I(Y))$. Hence, we conclude that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$, as desired.

b) In exercise 1.2, we showed that $I(Y) = (x^2 - y, x^3 - z)$. The hardest part of this exercise is to prove that one has produced a complete list of generators for $I(\overline{Y})$. I did not require this for full credit, since it is kind of tucked away in the problem statement, but if you're curious, I include a proof. It can be done systematically via the theory of Groebner bases, but what follows is a direct approach.

We need to start with some basic observations on the behavior of β. First, it is multiplicative. Second, it is almost additive:

$$\beta(f + g) = w^{-d_0}(w^{d-d_1}\beta(f) + w^{d-d_2}\beta(g)),$$

where $d = \max(\deg f, \deg g)$, $d_1 = \deg f$, $d_2 = \deg g$, and d_0 is maximal without introducing denominators. We see that $d - d_i = 0$ for at least one $i = 1, 2$, so $d_0 = 0$ unless there is cancellation of terms in the sum, and if $d_0 = 0$ we have $\beta(f + g)$ is in the ideal generated by $\beta(f), \beta(g)$. So in essense, the only problem is cancellation of the terms which are not multiples of w, which is equivalent to cancellation of highest degree terms in $f + g$, and only happens if $d_1 = d_2$.

We do see that this happens in our case: $I(Y)$ contains $x(x^2 - y) - (x^3 - z) = z - xy$, and also $(x^2 + y)(x^2 - y) - x(x^3 - z) = xz - y^2$. Then it is easy to check via degree considerations that neither $\beta(z - xy) = wz - xy$ nor $\beta(xz - y^2) = xz - y^2$ is in the ideal generated by $\beta(x^2 - y) = x^2 - wy$ and $\beta(x^3 - z) = x^3 - w^2 z$. We claim however that $I(\overline{Y}) = \beta(I(Y))$ is generated by $(x^2 - wy, wz - xy, xz - y^2)$.

We develop a few more observations on the behavior of β in general. Given polynomials $f_1, \ldots, f_m$ and $g_1, \ldots, g_m$, we are interested in when $\beta(\sum_i f_i g_i)$ is in the ideal generated by the $\beta(g_i)$. As above, we see this is only the case unless the highest degree terms of

f_ig_i cancel in the sum. If the $\beta(g_i)$ fail to generate the ideal $\beta(\sum_i f_ig_i)$, then there is some $f_1,\dots,f_m$ such that the maximum degree of the f_i is as small as possible, and $\beta(\sum_i f_ig_i)$ is not in the ideal generated by the $\beta(g_i)$. For convenience (and because it applies in our case) we assume that the g_i all have the same degree. Let $d = \deg(\sum_i f_ig_i)$. Write each $f_i = h_i + r_i$, where every term of h_i has degree strictly greater than $d - \deg g_i$, and and r_i has degree less than or equal to $d - \deg g_i$. Then $\sum_i f_ig_i = \sum_i h_ig_i + \sum_i r_ig_i$. The degree of $\sum_i r_ig_i$ is at most d, and the degree of $\sum_i f_ig_i$ is d by assumption, so we see that the degree of $\sum_i h_ig_i$ is at also at most d, and the top degree terms do not cancel when we have $\sum_i h_ig_i$ to $\sum_i r_ig_i$. Since the r_i have degree strictly smaller than the largest degree of the f_i, by our minimality hypothesis we conclude that $\beta(\sum r_ig_i)$ is in the ideal generated by the $\beta(g_i)$. Thus, if we can show that $\beta(\sum_i h_ig_i)$ is also in the ideal, since the highest degree terms do not cancel, we conclude that $\beta(\sum_i f_ig_i)$ is also in the ideal, contradicting our assumption that it was not.

Returning to our specific case, we have the following situation: we have h_1, h_2, h_3 with every term of degree strictly greater than $d-2$, but such that the degree of

$$h := h_1(x^2 - y) + h_2(z - xy) + h_3(xz - y^2)$$

is at most d. Our claim is that in this case, we actually have h a multiple of $xz - y^2$, so that it immediately follows that $\beta(h)$ is in the ideal generated by $\beta(xz - y^2)$, and we are done. The proof is by induction on the maximal degree d' of the h_i. The hypothesis that h has degree at most d implies that if we split $h_i = q_i + s_i$, with q_i homogeneous of degree d' (or $q_i = 0$) and s_i of degree strictly less than d', then the highest degree term of

$$q := q_1(x^2 - y) + q_2(z - xy) + q_3(xz - y^2)$$

cancels. This highest degree term is $q_1x^2 - q_2xy + q_3xz - q_3y^2$. Its cancellation means we have $q = q_2z - q_1y$. Moreover, if we solve for q_1 above we get $q_1 = \frac{1}{x^2}(q_2xy - q_3xz + q_3y^2)$, so

$$\begin{aligned} q &= q_2z - \frac{y}{x^2}(q_2xy - q_3xz + q_3y^2) \\ &= \frac{1}{x^2}(q_2x^2z - q_2xy^2 + q_3xyz - q_3y^3) \\ &= \frac{1}{x^2}(q_2x + q_3y)(xz - y^2). \end{aligned}$$

Since q is a polynomial, and x doesn't divide $xz - y^2$, we conclude that x^2 divides $(q_2x + q_3y)$, so q is a multiple of $xz - y^2$. Now we replace h_1 by s_1, h_2 by s_2, and h_3 by $s_3 + \frac{q_2x+q_3y}{x^2}$, all of which have degree strictly less than d' if they are non-zero, and we have not changed the value of h. If d' was the minimal value of $d-1$, then in fact all $s_i = 0$, and we have written h as a multiple of $xz - y^2$, as desired. If d' was not minimal, then we have reduced the degrees of the h_i without changing anything else, and we can repeat inductively.

3.1 b) Show that $\mathbb{A}^1$ is not isomorphic to any proper open subset of itself.
c) Any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.
d) We will see later (Ex. 4.8) that any two curves are homeomorphic. But show now that $\mathbb{A}^2$ is not even homeomorphic to $\mathbb{P}^2$.

Proof: b) Most people argued, perfectly reasonably, using the units of the ring of regular functions. Here is an alternate argument. Suppose $\mathbb{A}^1$ is isomorphic to U, an open subset of itself. Then there exists an invertible morphism $\varphi : \mathbb{A}^1 \to U$. Then since $U \subseteq \mathbb{A}^1$, we see φ is regular. Because $\mathcal{O}(\mathbb{A}^1) = A(\mathbb{A}^1) = k[x]$, it follows that φ is a polynomial in x. It must be nonconstant if it defines an isomorphism onto an open subset of $\mathbb{A}^1$. However, over an algebraically closed field, a nonconstant polynomial function always gives a surjective map, so $U \supseteq \varphi(\mathbb{A}^1) = \mathbb{A}^1$.

c) The basic idea is to use linear change of coordinates (which induces an automorphism on $\mathbb{P}^1$, and therefore doesn't affect the isomorphism class of the conic) to show that any two conics are isomorphic, and then to prove that a particular convenient conic is isomorphic to $\mathbb{P}^1$. One can work by brute force, but there is a more elegant alternative using the following lemma:

Lemma: Given any three points $P, Q, R \in \mathbb{P}^2$ not on a line, and any other three points $P', Q', R' \in \mathbb{P}^2$ also not a line, there is a linear change of coordinates sending (P, Q, R) to (P', Q', R').

Proof: a linear change of coordinate in $\mathbb{P}^2$ corresponds to a linear automorphism of k^3. Then P, Q, R and P', Q', R' correspond to lines in k^3 through the origin, and the hypothesis that they are not collinear is exactly saying that they do not lie in some plane, which is to say, each triple spans k^3. Choosing a non-zero vector in each line, we get two bases, and the corresponding change of basis map is what we want.

Using the lemma, if we choose any P, Q, R in the conic and not on a line, we have a change of coordinates sending them to $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, which then puts the conic in the form $Z(axy + bxz + cyz)$, where we see that all of a, b, c must be non-zero in order to have irreducibility. We claim we can scale (x, y, z) by some $(\lambda_1, \lambda_2, \lambda_3)$ respectively so that $a = b = c = 1$. Solving the resulting equations directly we find the solution $\lambda_1 = \sqrt{\frac{c}{ab}}$, $\lambda_2 = \sqrt{\frac{b}{ac}}$, $\lambda_3 = \sqrt{\frac{a}{bc}}$. It follows that any conic in $\mathbb{P}^2$ is isomorphic to $Z(xy + xz + yx)$, and any two conics are isomorphic.

In particular, any conic is isomorphic to $Z(y^2 - xz)$, so we will show that this is isomorphic to $\mathbb{P}^1$. We see that it is the image of the morphism $\mathbb{P}^1 \to \mathbb{P}^2$ given by $(u, v) \mapsto (u^2, uv, v^2)$ (note that this is defined everywhere because if u^2, uv, v^2 vanish simultaneously, then $u = v = 0$). To see that this inverse map is a morphism, we note that it is given by (x, y) away from $(0, 0, 1)$ (since $(u^2, uv) = (u, v)$ on $\mathbb{P}^1$ for $u \neq 0$) and by (y, z) away from $(1, 0, 0)$.

Remark: We can define morphisms between projective varieties using tuples of homogeneous polynomials as above because under our standard affine cover U_i of projective space, we will get maps defined by rational functions where we divide by the ith polynomial; this is thus regular away from the vanishing locus of the ith polynomial, and using the full cover we find it is regular except where all the polynomials vanish.

Remark: This exercise brings out the geometry in a), and shows again that compactified varieties are easier to work with: we see that an affine conic has projective closure isomorphic to $\mathbb{P}^1$. The reason there are two possibilities is that the closure can meet the line at infinity at two points, in which case the affine piece is isomorphic to $\mathbb{P}^1 \smallsetminus \{0, \infty\} = \mathbb{A}^1 \smallsetminus \{0\}$ or at one point (with multiplicity 2), in which case the piece is isomorphic to $\mathbb{P}^1 \smallsetminus \{\infty\} = \mathbb{A}^1$.

Remark: The lemma is actually true in a stronger form, since there is flexibility about which vectors we pick in each line.

d) We will first prove that any two curves in $\mathbb{P}^2$ have a nonempty intersection. [This is Exercise 3.7/(a), and although it is a later exercise, since the statement had been mentioned in class I did not require a proof to get full credit on the problem.] Suppose the curves $C_i \subseteq \mathbb{P}^2$ are defined by homogeneous polynomials f_i, for $i = 1, 2$. We may assume $C_1 \neq C_2$ as otherwise the claim is trivial. For each i, we denote by $\tilde{C}_i$ the affine variety $Z(f_i) \subseteq \mathbb{A}^3$. Let $Z = C_1 \cap C_2 = Z(f_1, f_2) \subseteq \mathbb{P}^2$ and $\tilde{Z} = \tilde{C}_1 \cap \tilde{C}_2 = Z(f_1, f_2) \subseteq \mathbb{A}^3$. Note that $\tilde{Z}$ contains the origin, and lines through the origin in $\tilde{Z}$ correspond to points of Z, so Z is non-empty if and only if $\tilde{Z}$ contains points other than the origin. By exercise 1.8, every component of $\tilde{Z}$ has dimension 1. But $\tilde{Z}$ contains the origin, so we conclude that it is non-empty of dimension 1, and thus that Z is non-empty, as desired.

This statement is not true for curves in $\mathbb{A}^2$. For example, any two parallel lines have empty intersection. Since dimension of closed subsets is defined entirely topologically, the notion of a curve is topological, so we conclude that $\mathbb{A}^2$ cannot be homeomorphic to $\mathbb{P}^2$.

3.9 The homogeneous coordinate ring of a projective variety is not invariant under isomorphism. For example, let $X = \mathbb{P}^1$, and let Y be the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$. Then $X \cong Y$ (Ex. 3.4). But show that $S(X) \not\cong S(Y)$.

Proof: $Y = \{(a^2, ab, b^2) \mid a, b \in k\}$. It is easy to see that $Y = Z(y^2 - xz)$, and to check directly that $z^2 - xy$ is irreducible in $k[x, y, z]$. Therefore, $I(Y) = (y^2 - xz)$, and $S(Y) = k[x, y, z]/(y^2 - xz)$. Many people argued that in $S(Y)$ we have $xz = y^2$, so it is not a unique factorization domain. This approach is fine, but more difficult than anyone realized: one has to check that x, y, z are really irreducible, and also that they are not unit multiples of one another. One can check these via the observation that any polynomial in $S(Y)$ can uniquely represented by a polynomial in the form of $f(x, z) + g(x, z)y$. However, the argument is non-trivial.

The following alternate argument turned out to be more involved than I expected, but it brings in some interesting ideas, so I'm including it just for fun. We claim that no homomorphism $S(Y) \to k[u, v]$ can be surjective, let alone an isomorphism. Given a homomorphism, the images of x, y, z are some polynomials f, g, h satisfying $fh = g^2$, and the image of the homomorphism is contained in the ring generated by f, g, h over k, since we know that constants are units and must be mapped to units, hence to constants. We claim that no such triple of polynomials can generate $k[u, v]$ over k. The basic observation we will use is that if $f_1, \dots, f_n \in k[u, v]$ generate the ideal I, then the subring generated by the f_i is contained in the subring $k + I$. Moreover, $k + I = k[u, v]$ if and only if $I = k[u, v]$ or $k[u, v]/I \cong k$, which is equivalent to I being maximal.

Now suppose that f, g, h have some non-constant common factor q, which we can assume without loss of generality to be irreducible. Then the ring they generate over k is contained in $k + (q)$, and we see this cannot be the entire ring: indeed, (q) has height 1 by the principal ideal theorem, so cannot be maximal. We may thus assume that f and h are relatively prime, so they are each then perfect squares, say $f = (f')^2$ and $h = (h')^2$, and then $g = f'h'$. Let $I \subseteq k[u, v]$ be the ideal generated by f', h'. There are two cases to consider. We first suppose $I \neq k[u, v]$. Then the ring generated by f, g, h is contained in $k + I^2$, so it suffices to show that $k + I^2 \subsetneq k[u, v]$. We claim that in fact this is true for any ideal $I \subsetneq k[u, v]$. Indeed, any such I is contained in some maximal ideal $\mathfrak{m}$, so $k + I^2 \subseteq k + \mathfrak{m}^2$, and it suffices to see that $\mathfrak{m}^2$ cannot be maximal. However, we argue directly: we know $\mathfrak{m}$ is of the form $(u - a, v - b)$ for some $a, b \in k[u, v]$. Then if we expand elements of $k[u, v]$ as polynomials in $u - a$ and $v - b$, it is clear that $\mathfrak{m}^2$ is precisely the set of polynomials with no constant or linear terms, and $k + \mathfrak{m}^2$ is then the set of polynomials with no linear terms, so is not all of $k[u,v]$.

It remains to check the case that $I = (f', h') = k[u,v]$. In this case, we consider the map $k[u,v] \to k[u,v]$ given by $u \mapsto f'$, $v \mapsto h'$; showing that f' and h' don't generate all of $k[u,v]$ over k is equivalent to showing that this map is not surjective. Now, we observe that any surjective map $k[u,v] \to k[u,v]$ is necessarily an isomorphism: if it is not injective, we can take the iterated preimage of 0 under the map, and we will get an infinite strictly increasing chain of ideals, whose union is an ideal which cannot be finitely generated. [This argument works for any Noetherian ring] Thus we see that to prove the map is not surjective, it suffices to prove it is not an isomorphism. But consider the induced morphism $\mathbb{A}^2 \to \mathbb{A}^2$. The hypothesis that $(f', h') = k[u,v]$ means $1 \in (f', h')$, which implies that f' and h' have no common zero. This means that $(0,0)$ is not in the image of the morphism $\mathbb{A}^2 \to \mathbb{A}^2$, which means that the morphism is not an isomorphism. Thus the ring map could not have been an isomorphism, and we conclude that f', h' cannot generate $k[u,v]$ over k.

Remark: Using singularity theory, we can now give a very short proof for this exercise. Indeed, for the rings to be isomorphic is equivalent to the corresponding affine varieties being isomorphic, but $\mathbb{A}^2$ is non-singular, while the Jacobian condition implies that $Z(xz - y^2) \subseteq \mathbb{A}^3$ is singular at the origin, so they cannot be isomorphic.