HOMEWORK 4
SELECTED SOLUTIONS

2.8 If X is a classical variety, and $U \subseteq X$ open, then $f : U \to k$ is regular in the above sense if and only if it is regular in the sense we defined for classical varieties.

Solution: This problem is actually straightforward once one carefully thinks through what we already know about regular functions and classical varieties. The two mains points are the following:

(i) If we have a quasiaffine variety Y, and consider it to be quasiprojective via one of the standard inclusions $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$, then the regular functions on open subsets of Y are the same whether we consider Y to be quasiaffine or quasiprojective.

(ii) If Y is affine, and $f \in A(Y)$, then $Y_f := Y \smallsetminus Z(f)$ is isomorphic to an affine variety, meaning regular functions on its open subsets are the same whether we consider it as affine variety or as an open subset of Y. We will abuse notation slightly and denote that affine variety by Y_f, so that the map $Y_f \to Y$ is a morphism of affine varieties giving an isomorphism onto its image $Y \smallsetminus Z(f)$.

Now let X be a projective variety, with atlas $\varphi_i : X_i \to U_i$. The atlas is obtained by intersecting with $\mathbb{P}^n \smallsetminus Z(x_j)$ and using the isomorphism $\mathbb{A}^n \cong \mathbb{P}^n \smallsetminus Z(x_i)$, and each φ_i is an isomorphism by (i) above. Given $U \subseteq X$ open, and $f : U \to k$, we then have that U is regular in the classical sense if and only if it is regular in the classical sense on each $U_i \cap U$ (because regularity in the classical sense is a local condition) if and only if it is regular in the classical sense on each $\varphi_i^{-1}(U)$ (because each φ_i is an isomorphism), which is the definition of f being regular in the prevariety sense.

Now we prove the following general lemma:

Lemma: If X is a prevariety, and $U \subseteq X$ open considered as a prevariety in the usual way, and $V \subseteq U$ open, then a function $f : V \to k$ is regular for $V \subseteq U$ if and only if it is regular for $V \subseteq X$.

Proof: Let $\varphi_i : X_i \to U_i$ be the atlas for X, and $\varphi_{i,j} : X_{i,j} \to U_{i,j}$ the atlas for U, with the notation indicating that for a fixed i, the $X_{i,j}$ are an affine cover of X_i of the form $(X_i)_{f_{i,j}}$, and $U_{i,j} = \varphi_i(X_{i,j})$, and $\varphi_{i,j}$ induced by φ_i and the isomorphism of $(X_i)_{f_{i,j}}$ onto $X_i \smallsetminus Z(f_{i,j})$ (using (ii) above). Now, by definition f is regular for $V \subseteq X$ if and only if $f \circ \varphi_i$ is regular (in the classical sense) on $\varphi_i^{-1}(V) \subseteq X_i$ for each i. Because regularity in the classical sense is local, and the each X_i is covered by the $X_i \smallsetminus Z(f_{i,j})$, this is equivalent to $f \circ \varphi_i$ being regular after restriction to $X_i \smallsetminus Z(f_{i,j})$ for all i, j. But since the maps $X_{i,j} \to X_i \smallsetminus Z(f_{i,j})$ are isomorphisms, and $\varphi_{i,j}$ is the composition of φ_i with these maps, this is equivalent to $f \circ \varphi_{i,j}$ being regular on $\varphi_{i,j}^{-1}(V)$ for all i, j, which is equivalent to f being regular on $V \subseteq U$, as desired.

Now, suppose X is quasiprojective, and $U \subseteq X$ open, and $f : U \to k$ a function. Let Y be the closure of X in projective space, so Y is projective. Then by definition f is regular in the classical sense for $U \subseteq X$ if and only if it is regular in the classical sense for $U \subseteq Y$, and by the above this is true if and only if it is regular in the prevariety sense for $U \subseteq Y$. By the lemma, this is equivalent to f being regular in the prevariety sense on $U \subseteq X$. This proves what we want for quasiprojective varieties.

The case of quasiaffine varieties is similar, but easier. One could deduce it from the quasiprojective case using (i) above by proving that the atlases we obtain considering as a quasiaffine variety as quasiaffine or as quasiprojective give the same open subsets, but it is easier to prove it directly. In the affine case, we have a one-element atlas, and regular functions are equivalent by definition. The quasiaffine case then follows by the lemma.

Remark: Some people asked whether they had to check this question for "any" atlas of a classical variety, and I remarked that we hadn't even defined what "any" atlas would mean. However, the above argument makes it clear what it should mean: every chart of the atlas should be an isomorphism of classical varieties.

2.10 If $\varphi : X \to Y$ is a morphism, and $U \subseteq X$ is an open subset considered as a prevariety, then $\varphi|_U$ is a morphism.

A continous map $\varphi : X \to Y$ is a morphism if and only if there is some open cover U_i of X such that $\varphi|_{U_i}$ is a morphism for each i.

Solution: The main point of both parts of this problems is that being a regular function is a local condition, when the open subsets are considered as prevarieties. Note that being a regular function is local for classical varieties immediately from the definitions, but this is not the case for prevarieties. Or more accurately, it is clear from the definition of regular function for a prevariety (together with the local nature in the classical case) that if $f : U \to k$ is a function for U open in X, and U_i an open cover of X, then f is regular on U if and only if its restriction to each $U \cap U_i$ is regular, when everything is considered as an open subset of X. It not immediately clear that it is true when $U \cap U_i$ is considered as an open subset of the prevariety U_i, but for this it is enough to know that a function on $U \cap U_i$ is regular when $U \cap U_i$ is considered as a subset of X if and only if it is regular when $U \cap U_i$ is considered as a subset of U_i, and this is precisely the content of the above lemma.

Now, for the first part the restriction of a continuous map is continuous, so we need only show that if $V \subseteq Y$ is open, and $f : V \to k$ is regular, then $f \circ \varphi : (\varphi|_U)^{-1}(V) \to k$ is regular as a function on $U \cap \varphi^{-1}(V) \subseteq U$. But $f \circ \varphi$ is regular on $\varphi^{-1}(V)$ by the hypothesis that $\varphi : X \to Y$ is a morphism, and our argument above then gives us the regularity of $f \circ \varphi : (\varphi|_U)^{-1}(V) \to k$ on $U \cap \varphi^{-1}(V)$ considered inside U, which proves the induced map $U \to Y$ is a morphism.

For the second part, in fact the continuity hypothesis is superfluous, since φ is continuous if and only if $\varphi|_{U_i}$ is continuous for each i. The "only if" direction follows immediately from the first part, so we need to see that if $\varphi|_{U_i}$ is a morphism for all i, then φ is a morphism. But let $f : V \to k$ be a regular function; then by hypothesis, $f \circ \varphi$ is regular on each $\varphi^{-1}(V) \cap U_i$ considered as an open subset of U_i, so by the above discussion we conclude it is regular on all of $\varphi^{-1}(V)$ (considered as an open subset of X), and thus φ is a morphism.