# HOMEWORK 5
# SELECTED SOLUTIONS

**4.3** a) Let $f$ be the rational function on $\mathbb{P}^2$ given by $f = x_1/x_0$. Find the set of points where $f$ is defined and describe the corresponding regular function.

b) Now think of this function as a rational map from $\mathbb{P}^2$ to $\mathbb{A}^1$. Embed $\mathbb{A}^1$ in $\mathbb{P}^1$ and let $\varphi : \mathbb{P}^2 \to \mathbb{P}^1$ be the resulting rational map. Find the set of points where $\varphi$ is defined, and describe the corresponding morphism.

Solution: a) $f$ is clearly defined on $U = \mathbb{P}^2 \smallsetminus Z(x_0)$. We claim it is not defined anywhere on $Z(x_0)$. Given $P \in Z(x_0)$, if $f$ is defined at $P$, there is some neighborhood $V$ of $P$ and homogeneous polynomials $g, h$ of the same degree such that $V \cap Z(h) = \emptyset$, and $\frac{x_1}{x_0} = \frac{g}{h}$ on $U \cap V$. Observe that we cannot have $x_0 | h$, since $Z(h) \cap V = \emptyset$, and $P \in Z(x_0)$. Now, $U \cap V \subseteq U$, and we can identify $U$ with $\mathbb{A}^2$ by setting $x_0 = 1$. We then find that on $U \cap V \subseteq \mathbb{A}^2$, we have $x_1 = \frac{g(1,x_1,x_2)}{h(1,x_1,x_2)}$, so $h(1, x_1, x_2)x_1 = g(1, x_1, x_2)$ on $U \cap V$. But by injectivity of restriction to open subsets, we conclude this identity holds on all of $\mathbb{A}^2$, so the two polynomials must agree. But this is not possible: since $x_0$ cannot divide $h$, the degree of $h(1, x_1, x_2)$ is equal to the degree of $h$, so the degree of the left side is strictly larger than the degree of the right side.

b) Away from $Z(x_0)$, we have that $\varphi$ is given by

$$(x_0, x_1, x_2) \mapsto (1, \frac{x_1}{x_0}) = (x_0, x_1).$$

This visibly extends to a morphism except when $x_0 = x_1 = 0$. We claim that $\varphi$ cannot extend over $(0, 0, 1)$. One can argue algebraically as above, but for the sake of variety, we give a geometric argument. For each $c$, consider the line through $(0, 0, 1)$ given by $Z(x_0 - cx_1)$. We can parametrize (an open subset of) this line by the morphism $\varphi_c : \mathbb{A}^1 \to \mathbb{P}^2$ given by $t \mapsto (ct, t, 1)$. Now, away from $t = 0$ we have the composition $\varphi \circ \varphi_c$ which sends $t$ to $(ct, t) = (c, 1)$ for all $t$, so if $\varphi$ extends to a morphism over $(0, 0, 1)$, we see that it must send $(0, 0, 1)$ to $(c, 1)$ as well. But this has to hold for all $c$, which isn't possible.

**4.7** Let $X$ and $Y$ be two varieties. Suppose there are points $P \in X$ and $Q \in Y$ such that the local rings $\mathcal{O}_{P,X}$ and $\mathcal{O}_{Q,Y}$ are isomorphic as $k$-algebras. Then show that there are open sets $P \in U \subseteq X$ and $Q \in V \subseteq Y$ and an isomorphism of $U$ to $V$ which sends $P$ to $Q$.

Solution: First let $U'$ and $V'$ be any affine open neighborhoods of $P$ and $Q$, and $\varphi_P^* : \mathcal{O}_{Q,Y} \to \mathcal{O}_{P,X}$ an isomorphism. We have injections $A(U') \hookrightarrow \mathcal{O}_{P,X}$ and $A(V') \hookrightarrow \mathcal{O}_{Q,Y}$, and $\mathcal{O}_{P,X} \subseteq K(A(U'))$. So we get an induced homomorphism $A(V') \to K(A(U'))$, whose image does not necessarily lie in $A(U')$. However, we know that $A(V')$ is a finitely-generated $k$-algebra, so let $y_1, \dots, y_m$ be generators. Then $\varphi_P^*(y_i) = \frac{x_i}{z_i}$ for some $x_i, z_i \in A(U')$, and moreover we may require $z_i(P) \neq 0$ since $\varphi_P^*$ maps into $\mathcal{O}_{P,X}$. Let $h = \prod_i z_i$. Then $P \notin Z(h)$, so we have $(U')_h$ an open neighborhood of $P$, and we find that $\varphi_P^*$ gives us a homomorphism

$$\varphi^* : A(V') \to A((U')_h) = A(U')_h.$$

This corresponds to a morphism $\varphi : (U')_h \to V'$ of affine open subvarieties of $X$ and $Y$.

We next claim that $\varphi$ maps $P$ to $Q$. If $\mathfrak{m}_P$ and $\mathfrak{m}_Q$ are the maximal ideals of $A((U')_h)$ and $A(V')$ corresponding to $P$ and $Q$ respectively, then $\mathfrak{m}'_P := \mathfrak{m}_P \mathcal{O}_{P,X}$ and $\mathfrak{m}'_Q := \mathfrak{m}_Q \mathcal{O}_{Q,Y}$ are the maximal ideals of the local rings. Now, we know that $(\varphi_P^*)^{-1}(\mathfrak{m}'_P) = \mathfrak{m}'_Q$, since $\varphi_P^*$ is an isomorphism of local rings and must therefore indentify their maximal ideals. We conclude that $(\varphi^*)^{-1}(\mathfrak{m}_P) \subseteq \mathfrak{m}_Q$, but we also know that $(\varphi^*)^{-1}(\mathfrak{m}_P)$ is the maximal ideal corresponding to $\varphi(P)$, so it must be all of $\mathfrak{m}_Q$, and we must have $\varphi(P) = Q$.

Now, by the same argument with the roles of $X$ and $Y$ reversed we obtain a morphism $(V')_{h'} \to U'$ induced by $(\varphi_P^*)^{-1}$, and the compositions are the identity where they are defined, so restricting sets of definition as in the proof of Corollary I.4.5, we conclude that $\varphi$ gives an isomorphism on open neighborhoods of $P$ and $Q$, as desired.