HOMEWORK 6
SELECTED SOLUTIONS

3.2 Show that if X is a variety, and for some $P, Q \in X$ we have $\mathcal{O}_{P,X} \subseteq \mathcal{O}_{Q,X} \subseteq K(X)$, then $P = Q$.

Solution: Let U and V be affine open neighborhoods of P, Q, respectively. Then $A(U) \subseteq \mathcal{O}_{P,X}$ and $A(V) \subseteq \mathcal{O}_{Q,X}$. As in Exercise I.4.7 of Hartshorne, the inclusion $\mathcal{O}_{P,X} \hookrightarrow \mathcal{O}_{Q,X}$ induces an injection $A(U) \to A(V_h) = A(V)_h$ for an appropriately chosen h. This gives a morphism $\varphi : V_h \to U$. We claim that $\varphi(Q) = P$. Indeed, let $\mathfrak{m}_P$ and $\mathfrak{m}_Q$ be the maximal ideals of $A(U)$ and $A(V_h)$ corresponding to P and Q respectively. Then $\mathfrak{m}'_P = \mathfrak{m}_P\mathcal{O}_{P,X}$ and $\mathfrak{m}'_Q = \mathfrak{m}_Q\mathcal{O}_{Q,X}$ are the maximal ideals of the local rings. Unlike in the Hartshorne exercise, we don't know *a priori* that the preimage of $\mathfrak{m}'_Q$ in $\mathcal{O}_{P,X}$ (i.e., $\mathfrak{m}'_Q \cap \mathcal{O}_{P,X}$) is equal to $\mathfrak{m}'_P$. However, we do see immediately that $\mathfrak{m}'_Q \cap \mathcal{O}_{P,X} \subseteq \mathfrak{m}'_P$, since anything not in $\mathfrak{m}'_P$ is a unit in $\mathcal{O}_{P,X}$ and hence in $\mathcal{O}_{Q,X}$. It follows that

$$\mathfrak{m}_Q \cap A(U) = \mathfrak{m}'_Q \cap A(V_h) \cap A(U) \subseteq \mathfrak{m}'_Q \cap \mathcal{O}_{P,X} \cap A(U) \subseteq \mathfrak{m}'_P \cap A(U) = \mathfrak{m}_P.$$

But $\mathfrak{m}_Q \cap A(U)$ is necessarily the maximal ideal corresponding to $\varphi(Q) \in U$, so the containment implies equality, and Q maps to P as desired.

Finally, we note that φ induces the identity on $K(X)$, so if we think of it as a morphism $V_h \to X$ inducing a rational map on X, then $\langle V_h, \varphi\rangle$ is equivalent to $\langle X, \mathrm{id}\rangle$ in the set of rational maps. By definition of the equivalence relation, we have $\varphi = \mathrm{id}$, so we conclude that $P = \varphi(Q) = Q$, as desired.

3.3 (a) Show that if $\varphi : X \to Y$ is a morphism of varieties, and $U \subseteq X$ is an open subset such that the composition $U \to Y$ is an isomorphism, then $U = X$.

(b) Show that if $\varphi : X \to Y$ is a morphism of varieties, and $U \subseteq X$ is an open subset such that $\varphi : U \to Y$ is an isomorphism onto an open subset $V \subseteq Y$, then $\varphi^{-1}(V) = U$.

Solution: (a) We have $\varphi^{-1} : Y \to U$, so if we compose $\varphi^{-1} \circ \varphi$ with the inclusion $U \hookrightarrow X$, we obtain a morphism $\psi : X \to X$ which agrees with the identity map after restricting to U. Since X is a variety, we conclude that ψ is the identity map, and since it factors through $U \hookrightarrow X$, we conclude $U = X$.

Remark: Note that the statement is false for prevarieties; we can let X be the affine line with the doubled origin, and U and Y the affine line.

(b) Consider the induced morphism $\varphi^{-1}(V) \to V$. We have $U \subseteq \varphi^{-1}(V)$ an open subset, which maps isomorphically onto V, so by (a) we have $U = \varphi^{-1}(V)$.

6.7 Let $P_1, \ldots, P_r, Q_1, \ldots, Q_s$ be distinct points of $\mathbb{A}^1$. If $\mathbb{A}^1 \smallsetminus \{P_1, \ldots, P_r\}$ is isomorphic to $\mathbb{A}^1 \smallsetminus \{Q_1, \ldots, Q_s\}$, show that $r = s$. Is the converse true?

Solution: By our theorem on extending morphisms from nonsingular curves to projective curves, an isomorphism has to extend to a morphism $\mathbb{P}^1 \to \mathbb{P}^1$, which also has to be an isomorphism since the inverse likewise extends. By bijectivity (on $\mathbb{P}^1$ as well as on the original open subsets) we find that we must have the P_i together with ∞ mapped to the Q_i together with ∞, so $r = s$.

The converse is not true. It is true if $r \leq 2$, but not for larger r. Indeed, if we choose $P_1 = 0, P_2 = 1, P_3 = \lambda$, and $Q_1 = 0, Q_2 = 1, Q_3 = 2$, we will show that there are only finitely many values of λ for which we can have an isomorphism. As above, an isomorphism would have to extend to an automorphism of $\mathbb{P}^1$. By 6.6, every such automorphism is obtained by linear change of variable.

We claim that such automorphisms are uniquely determined by where they send an ordered triple of points. Indeed, if v_1, v_2, v_3 and w_1, w_2, w_3 are two triples of non-zero vectors in k^2 such that no two of the v_i and no two of the w_i are scalar multiples of one another, then if we require that a linear transformation T send v_1 to a scalar multiple of w_1, and v_2 to a scalar multiple of w_2, we see that T is uniquely determined up to multiplication by an invertible diagonal matrix. But if $v_3 = c_1 v_1 + c_2 v_2$, we have that $T(v_3) = c_1 \lambda_1 w_1 + c_2 \lambda_2 w_2$, with the λ_i determined uniquely by the choice of diagonal matrix. If $w_3 = d_1 w_1 + d_2 w_2$, we have that the matrix is uniquely determined up to scalar by the condition that $T(v_3)$ be a scalar multiple of w_3. But scaling the matrix won't change its action on $\mathbb{P}^1$, so this proves the claim.

Now, an automorphism of $\mathbb{P}^1$ sending $\{0, 1, \lambda, \infty\}$ to $\{0, 1, 2, \infty\}$ will have to induce a particular bijection between these two sets of points. But by the claim, for any fixed choice of bijection, the automorphism of $\mathbb{P}^1$ is determined uniquely by where it sends 0, 1, and ∞, so then there is only one possible choice for the λ to get sent to the remaining point. Thus, there are only finitely many λ allowing an isomorphism, and we see that the converse is not true.

Remark: In fact, the cross ratio of four points on $\mathbb{P}^1$ is an invariant of automorphisms, so we can use it to easily check whether any given pair of triples on $\mathbb{A}^1$ give isomorphic curves.