HOMEWORK 8

The first two exercises are from the *Chevalley's theorem and complete varieties* lecture notes.

2.13 Chow's lemma. It is clear that every complete variety is birational to some projective variety. However, a much stronger statement is true: given a complete variety X, there exists a projective variety X' together with a birational morphism $X' \to X$ (which is necessarily surjective, by Corollary 2.11).

Let $\{U_i\}$ be an affine open cover of X, and Y_i be the closures of the U_i in projective space. Let U be the intersection of the U_i, and

$$\varphi : U \to X \times Y_1 \times \cdots \times Y_n$$

the morphism induced by the inclusion of U into X and the Y_i. Let X' be the closure of $\varphi(U)$. Let $p_1 : X' \to X$ be the morphism induced by the first projection, and $p_2 : X' \to Y_1 \times \cdots \times Y_n$ be the morphism induced by projection to the remaining factors.

a) Show that p_1 gives an isomorphism $p_1^{-1}(U) \to U$.
b) Show that p_2 induces an isomorphism of X' onto a closed subvariety of $Y_1 \times \cdots \times Y_n$.
c) Conclude Chow's lemma.

Proof: a) We first show that $\varphi(U)$ is open in X'. Consider $\varphi(U)$ as a subset of $U \times Y_1 \times \cdots \times Y_n$. Then $\varphi(U)$ is the graph of $p_2 \circ \varphi : U \to Y_1 \times \cdots Y_n$, so $\varphi(U)$ is the preimage of $\Delta(Y_1 \times \cdots \times Y_n)$ under the morphism

$$(p_2 \circ \varphi) \times id : U \times (Y_1 \times \cdots \times Y_n) \to (Y_1 \times \cdots \times Y_n) \times (Y_1 \times \cdots \times Y_n),$$

and we conclude that $\varphi(U)$ is closed in $U \times Y_1 \times \cdots \times Y_n \subseteq X \times Y_1 \times \cdots \times Y_n$. It follows that

$$\varphi(U) = X' \cap (U \times Y_1 \times \cdots \times Y_n).$$

However, $U \times Y_1 \times \cdots \times Y_n$ is open in $X \times Y_1 \times \cdots \times Y_n$, so $\varphi(U)$ is open in X'.

Now $p_1|_{\varphi(U)}$ is an isomorphism onto the open subset U of X. Then by part b) of Exercise 3.3 from the *Nonsingular curves* lecture notes, we have that $p_1^{-1}(U) = \varphi(U)$. Thus, p_1 gives an isomorphism $p_1^{-1}(U) \to U$.

b) Since X is complete and X' is closed, we have that $p_2(X')$ is closed in $Y_1 \times \cdots \times Y_n$. X' is irreducible, so $p_2(X')$ is irreducible. Hence, $p_2(X')$ is a closed subvariety of $Y_1 \times \cdots \times Y_n$. It suffices to show that p_2 induces an isomorphism $X' \to p_2(X')$.

We claim that for each i, we have

$$X'_i := X' \cap (U_i \times Y_1 \times \cdots \times Y_n) = X' \cap (X \times Y_1 \times \cdots \times U_i \times \cdots \times Y_n),$$

and moreover, if $(x, y_1, \ldots, y_n) \in X'$ has $x \in U_i$, then $x = y_i$. By definition, $p_1(X'_i) \subseteq U_i$. We can thus consider the composed morphism $X'_i \xrightarrow{p_1} U_i \to Y_i$. But we also have the projection morphism $X'_i \to Y_i$, and these morphisms agree on $\varphi(U) \subseteq X'_i$, which we know to be open by a). Thus these two morphisms agree on all of X'_i, so the projection of X'_i to Y_i has image contained in U_i, and we conclude that

$$X'_i \subseteq X' \cap (X \times Y_1 \times \cdots \times U_i \times \cdots \times Y_n),$$

and also that if $(x, y_1, \ldots, y_n) \in X'$ has $x \in U_i$, then $x = y_i$. To conclude the opposite inclusion, we argue similarly, using that the image of $X' \cap (X \times Y_1 \times \cdots \times U_i \times \cdots \times Y_n)$

under the projection to Y_i is contained in U_i, and composing with the inclusion $U_i \to X$. This completes the proof of the claim.

Now, $p_2 : X' \to p_2(X')$ is surjective by definition, and we now see that it is also injective: given $(x, y_1, \dots, y_n), (x', y_1, \dots, y_n) \in X'$, we have $x \in U_i$ for some i; then $y_i \in U_i$ also, and $x = y_i$. But then we also conclude that $x' \in U_i$, and that $x' = y_i$, giving injectivity. It thus suffices to see that p_2^{-1} is a morphism. But we see from the claim that

$$p_2(X'_i) = p_2(X') \cap Y_1 \times \cdots \times U_i \times \cdots \times Y_n,$$

so is an open subset of $p_2(X')$. Moreover, the $p_2(X'_i)$ cover $p_2(X')$, since the X'_i cover X'. But on $p_2(X'_i)$, the inverse map $p_2^{-1} : p_2(X'_i) \to X'$ is simply given by the morphism

$$Y_1 \times \cdots \times U_i \times \cdots \times Y_n \to X \times Y_1 \times \cdots \times Y_n$$

induced by the morphisms

$$Y_1 \times \cdots \times U_i \times \cdots \times Y_n \to U_i \to X$$

and

$$Y_1 \times \cdots \times U_i \times \cdots \times Y_n \to Y_1 \times \cdots \times Y_n.$$

Thus we see that $p_2 : X' \to p_2(X')$ is an isomorphism, as desired.

c) Since $Y_1 \times \cdots \times Y_n$ is a projective variety, $X' \cong p_2(X')$ is a projective variety. But by part a), X' is birational to X. Thus, we conclude Chow's lemma.

2.14 Prove that if X is complete variety, then $\mathcal{O}(X) = k$.

Proof: Two proofs: according to Chow's lemma, there is a projective variety X' with a surjective morphism $X' \to X$. Any regular function on X pulls back to a regular function on X', and must be constant on X' since we know $\mathcal{O}(X') = k$ for projective varieties. Then the function was constant on X by surjectivity of the morphism.

Alternatively, let $\varphi : X \to k$ be a regular map. We can consider it as a morphism $X \to \mathbb{A}^1$. This induces a morphism $\varphi : X \to \mathbb{P}^1$. By Proposition 2.4 from the notes, φ is a closed map. So $\varphi(X)$ is a irreducible closed subset of $\mathbb{P}^1$. The only irreducible closed subsets of $\mathbb{P}^1$ are one point or $\mathbb{P}^1$ itself. However, $\varphi(X) \subseteq \mathbb{A}^1$ cannot be $\mathbb{P}^1$. Thus, $\varphi(X)$ is one point, and φ is constant.

The next exercise is from the *Divisors on nonsingular curves* lecture notes.

1.8 Show that a nonconstant morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is ramified at all points of $\mathbb{P}^1$ if and only if it factors through the Frobenius morphism.

Proof: Thinking of $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$, with coordinate x on $\mathbb{A}^1$, there exist polynomial $f(x), g(x)$ without common factors such that $\varphi(x) = \frac{f(x)}{g(x)}$. We observe that φ factors through the Frobenius morphism if and only if f and g may both be expressed as polynomials in x^p.

Let $x = \lambda$ be a point in $\mathbb{A}^1$; then if $g(\lambda) \neq 0$, we have that $x - \varphi(\lambda) = x - \frac{f(\lambda)}{g(\lambda)}$ is a local coordinate at $\varphi(\lambda)$. Then

$$\varphi^*(x - \varphi(\lambda)) = \frac{f(x)}{g(x)} - \frac{f(\lambda)}{g(\lambda)} = \frac{f(x)g(\lambda) - f(\lambda)g(x)}{g(\lambda)g(x)}.$$

Since $g(\lambda) \neq 0$, the order of vanishing at λ is simply the order of vanishing of the numerator, so φ is ramified at λ if and only if $f(x)g(\lambda) - f(\lambda)g(x)$ has multiple roots at λ, if and only if its derivative $f'(x)g(\lambda) - f(\lambda)g'(x)$ has a root at λ.

Now, suppose φ is ramified everywhere. Then we have $f'(\lambda)g(\lambda) - f(\lambda)g'(\lambda) = 0$ for all λ with $g(\lambda) \neq 0$. This is an infinite number of values, so we conclude that

$$f'(x)g(x) = f(x)g'(x)$$

as polynomials. But f and g were assumed to have no common factors, and if f', g' are nonzero they have degree strictly less than f and g, so this is not possible unless $f' = g' = 0$, which means that f and g are polynomials in x^p, and φ factors through Frobenius.

For the converse, one could make two different arguments. One can argue directly that by the above, if φ factors through Frobenius, then $f' = g' = 0$, so φ is ramified at all points λ such that $g(\lambda) \neq 0$. The points at which $g(\lambda) = 0$ are those mapping to ∞, and $1/x$ is a local coordinate at ∞, with $\varphi^*(1/x) = \frac{g(x)}{f(x)}$. Thus the ramification index is the order of vanishing of $g(x)$ at λ, which is a multiple of p if g is a polynomial in x^p. Finally, one can similarly check that ∞ is ramified by choosing an appropriate local coordinate depending on whether or not $\varphi(\infty) = \infty$.

Alternatively, we see immediately that the Frobenius map itself is ramified at all points, since it is injective but has degree p, and the sum of the ramification indices over any point has to add up to the degree. It is then clear from the definitions that if we have a composition of two morphisms, and one is ramified at all points, then the composition is also ramified at all points.