ABSTRACT VARIETIES VIA ATLASES

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In this expository note we describe how abstract algebraic varieties over an algebraically closed field may be defined rigorously via an atlas definition analogous to the usual definition of differential manifolds. The restriction to algebraically closed fields allows us to use the naive notions of the underlying spaces and their morphisms; we thus fix throughout an algebraically closed field k.

1. The local models

In the definition of manifolds, one imposes a condition that every point have a neighborhood which is isomorphic (in some appropriate sense) to an open subset of $\mathbb{R}^n$. This is not possible for varieties, for two reasons. The first is that we do not want to restrict ourselves to smooth varieties, so even for complex varieties we will not necessarily obtain topological manifolds. However, even if we wished to restrict to smooth varieties, we could still not ask for our varieties to be locally isomorphic to an open subset of affine space, because algebraic maps are fundamentally much more rigid than differentiable maps, and it is simply not the case that a smooth variety has an open cover by varieties which can be thought of as open subvarieties of afffine space. We therefore allow our varieties instead to be covered by open subsets which are isomorphic to affine varieties.

Asking for a variety to have an open cover by affine varieties may at first be counterintuitive – in analogy with manifold theory, why not allow the open subsets to be isomorphic to open subsets of affine varieties? *A priori*, this is a reasonable question, as it is certainly possible for an open subset of an affine variety not to be isomorphic to an affine variety (see Exercise 3.6 of Chapter I of [1]). However, in fact it is not a concern, due to the following fact.

Lemma 1.1. *Let X be an affine variety. If $f \in \mathcal{O}(X)$, then $X_f := X \smallsetminus Z(f)$ is isomorphic to an affine variety, and the set of all X_f form a base for the Zariski topology on X.*

Proof. Indeed, we claim that if $X = Z(f_1, \dots, f_m) \subseteq \mathbb{A}^n$ for $f_i \in k[x_1, \dots, x_n]$, and $\tilde{f} \in k[x_1, \dots, x_n]$ any representative of f, then

$$X_f \cong Z(f_1, \dots, f_m, x_{n+1}\tilde{f} - 1) \subseteq \mathbb{A}^{n+1}.$$

The morphism $\mathbb{A}^{n+1} \to \mathbb{A}^n$ inducing the map to X_f is the projection onto the first n coordinates; the image is contained in X_f because is $x_{n+1}\tilde{f} - 1 = 0$ at a point, then $\tilde{f} \neq 0$. The inverse morphism sends $(x_1, \dots, x_n)$ to $(x_1, \dots, x_n, 1/\tilde{f})$, which is clearly an inverse and is defined on $\mathbb{A}^n \smallsetminus Z(\tilde{f})$. This proves the desired isomorphism.

It remains to see that the set of all X_f forms a base for the topology of X. But an open subset U of X is $X \smallsetminus Z(I)$, for some ideal $I \supseteq I(X)$. Given $P \in U$, by the ideal-variety correspondence we have $I(P) \not\supseteq I$, so choose $f \in I$ which is not in $I(P)$. then $Z(f) \supseteq Z(I)$, so $X_f \subseteq U$, and $P \in X_f$ since $f \notin I(P)$. Thus, the X_f form a base of the topology. □

It is thus sensible to conserve words by requiring our varieties to have open covers by affine varieties, since this is automatically satisfied if they have open covers by open subsets of affine varieties.

2. Prevarieties and their morphisms

General abstract varieties are obtained by gluing together affine varieties along open subsets, with the restriction that the gluing maps must be algebraic. We have:

Definition 2.1. A **prevariety** X over k is an irreducible topological space, together with an open cover $U_1, \ldots, U_m$, and a collection of homeomorphisms $\varphi_i : X_i \xrightarrow{\sim} U_i$, where each $X_i \subseteq \mathbb{A}^{n_i}$ is an affine variety equipped with the Zariski topology, and we require that every **transition map**

$$\varphi_i^{-1}(U_i \cap U_j) \overset{\varphi_j^{-1}\circ\varphi_i}{\xrightarrow{\sim}} \varphi_j^{-1}(U_i \cap U_j)$$

is a morphism. We say that each map $\varphi_i : X_i \to U_i \subseteq X$ is a **chart**, and the collection of charts is an **atlas**.

Remarks 2.2. Since the inverse of a transition map is a transition map, the transition maps are necessarily isomorphisms. One often thinks of a prevariety as being obtained from the collection of affine varieties X_i by gluing together open subsets along the isomorphisms given by the transition maps.

We have not yet defined varieties because we haven't yet imposed the condition analogous to the Hausdorff condition for a manifold. We will revisit this shortly.

One can vary the definition a bit by defining a notion of equivalence of atlases and speaking of a prevariety as a set with an equivalence class of atlases, or alternatively, by requiring an atlas to be maximal. Either of these options removes the "dependence on choice" of the atlas, but at this point it is not clear whether it would be any less technical to simply do what modern algebraic geometers do, which is to work with sheaves.

Example 2.3. Any affine variety is a prevariety, with an atlas consisting of a single chart. Any projective variety is a prevariety, since we have seen that projective space is covered by open subsets isomorphic to affine space.

Lemma 2.4. *Any open subset of a prevariety has a natural structure of a prevariety. Any irreducible closed subset of a prevariety has a natural structure of a prevariety.*

Proof. By Lemma 1.1, if X is a prevariety with a chosen atlas, and $U \subseteq X$ is open, then we can cover each $\varphi_i^{-1}(U_i \cap U) \subseteq X_i$ with affine open subsets (and indeed finitely many, since X_i is a Noetherian topological space), and passing to this refined cover and using the charts induced by the original φ_i, we get an atlas for U. In the closed case, since any irreducible closed subset of X_i is again an affine variety, we don't even need to refine the covers. □

Combining the example with the first part of the lemma, we conclude:

Corollary 2.5. *Any classical variety is a prevariety.*

The lemma also motivates the definition:

Definition 2.6. If X is a prevariety, a **subprevariety** of X is an irreducible closed subset of an open subset of X.

It follows that a subprevariety has the structure of a prevariety. One checks that a subprevariety of a subprevariety is a subprevariety, and it then follows that the classical varieties are precisely the subprevarieties of projective space.

We can also define morphisms of prevarieties, and in particular we will see when two prevarieties are isomorphic, helping us remove the dependence on choice of atlas. Following the approach of [1] for classical varieties, we first define regular functions.

Definition 2.7. If X is a prevariety with a given atlas, and $U \subseteq X$ is open, a function $f : U \to k$ is **regular** if for all i, the induced function

$$f \circ \varphi_i : \varphi_i^{-1}(U \cap U_i) \to k$$

is regular (in the sense defined for quasiaffine varieties).

It is not immediately obvious from the definitions, but in fact this concept of regular function is completely compatible with the definition for classical varieties, and also with our concept of subprevarieties. These compatibilities are laid out in the exercises that follow.

Exercise 2.8. If X is a prevariety with atlas $\{\varphi_i : X_i \to U_i\}$, and $U \subseteq U_i$, then a function $f : U \to k$ is regular if and only if $f \circ \varphi_i$ is regular in the classical sense on $\varphi_i^{-1}(U) \subseteq X_i$.

Exercise 2.9. If X is a prevariety, and $V \subseteq U \subseteq X$ open subsets of X, a function $f : V \to k$ is regular when V is considered as an open subset of X if and only if f is regular when V is considered as an open subset of U, and U is considered as a prevariety.

Exercise 2.10. If X is a prevariety, and $\{U_i\}$ is an open cover of X, and we consider each U_i as a prevariety, then for any open subset $V \subseteq X$, and function $f : V \to k$, we have that f is regular if and only if for each i we have $f|_{U_i \cap V}$ is regular when $U_i \cap V$ is considered inside U_i.

Exercise 2.11. If X is a classical variety, and $U \subseteq X$ open, then $f : U \to k$ is regular in the above sense if and only if it is regular in the sense we defined for classical varieties.

We then define morphism as usual:

Definition 2.12. Given prevarieties X, Y with atlases given by $\{\varphi_i : X_i \xrightarrow{\sim} U_i\}_i$ and $\{\psi_j : Y_j \xrightarrow{\sim} V_j\}_j$, a **morphism** $\varphi : X \to Y$ is a continuous map such that for all $U \subseteq Y$ open, and all $f : U \to k$ regular, we have $f \circ \varphi : \varphi^{-1}(U) \to k$ is also regular.

It is immediate from Exercise 2.11 that if X, Y are classical varieties, morphisms $X \to Y$ in the above sense are the same as morphisms in the sense we had already defined. We also see that compositions of morphisms are morphisms. We next see that the condition of being a morphism is a local one.

Exercise 2.13. If $\varphi : X \to Y$ is a morphism, and $U \subseteq X$ is an open subset considered as a prevariety, then $\varphi|_U$ is a morphism.

A map $\varphi : X \to Y$ is a morphism if and only if there is some open cover U_i of X such that $\varphi|_{U_i}$ is a morphism for each i.

Morphisms are also local on the target.

Exercise 2.14. If $\varphi : X \to Y$ is a morphism, and $V \subseteq Y$ is an open subset of Y, if we consider V and $\varphi^{-1}(V)$ as prevarieties, then φ induces a morphism $\varphi^{-1}(V) \to V$.

A continuous map $\varphi : X \to Y$ is a morphism if and only if there is some open cover V_i of Y such that φ induces a morphism $\varphi^{-1}(V_i) \to V_i$ for each i.

A definition of morphism more analogous to a typical one using atlases for differentiable manifolds is then the following:

Exercise 2.15. With notation as in the above definition, a continous map $\varphi : X \to Y$ is a morphism if and only if for any i, j, the induced map

$$(\varphi_i)^{-1}(\varphi^{-1}(V_j)) \xrightarrow{\varphi_i} \varphi^{-1}(V_j) \cap U_i \xrightarrow{\varphi} V_j \xrightarrow{(\psi_j)^{-1}} Y_j$$

is a morphism of quasiaffine varieties.

Following are some basic properties of morphisms.

Exercise 2.16. If $U \subseteq X$ is open, the inclusion map of prevarieties is a morphism.

If $Z \subseteq X$ is closed and irreducible, the inclusion map of prevarieties is a morphism.

If X and Y are prevarieties, Z a subprevariety of Y, and $\varphi : X \to Y$ any map with $\varphi(X) \subseteq Z$, then φ is a morphism if and only if the induced map $X \to Z$ is a morphism.

3. Abstract varieties

Note that a prevariety X is never Hausdorff (unless X consists of a single point), since it is irreducible by hypothesis. However, the analogue of the Hausdorff condition for a manifold is precisely what is missing from our definition. It turns out that the right definition involves the following fact from point-set topology:

Exercise 3.1. A topological space X is Hausdorff if and only if the image of the diagonal map $X \to X \times X$ is closed.

We will use the same definitions for varieties, but because the Zariski topology on a product of varieties is not the product topology, we will obtain a different and better-behaved notion.

Of course, we first need to define the product of prevarieties.

Definition 3.2. Given prevarieties X, Y, we define the **product** $X \times Y$ of X with Y to be the product set $X \times Y$, equipped with the atlas

$$\varphi_i \times \varphi_j : X_i \times Y_j \xrightarrow{\sim} U_i \times V_j,$$

and the topology induced by the atlas.

Exercise 3.3. The above definition gives a valid prevariety.

If X and Y are affine, this definition is consistent with the one we already have (and used in the above) for products of affine varieties.

If $Y \subseteq X$ is a subprevariety, then the topology on $Y \times Y \subseteq X \times X$ is the subset topology.

We then have that any prevariety X has a natural diagonal morphism:

Proposition 3.4. *Given a prevariety X, the diagonal map $\Delta : X \to X \times X$ is a morphism of prevarieties.*

Proof. Because it is enough to check locally whether or not a map is a morphism, we can take an atlas $\{\varphi_i : X_i \to U_i\}$ of X and check whether the induced map $U_i \to X \times X$ is a morphism for each i. The image of this map is contained in $U_i \times U_i$, so we reduce to checking whether $U_i \to U_i \times U_i$ is a morphism, and hence we have reduced the proposition to the case that X is affine. But in this case it is clear, as if $X \subseteq \mathbb{A}^n$, the map $X \to X \times X$ is induced by the polynomial map $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, a_1, \dots, a_n)$. $\square$

Our analogy to the Hausdorff condition is then the following:

Definition 3.5. We say that a prevariety X is a **variety** if the image of the diagonal morphism is closed.

Example 3.6. Any affine variety is a variety. Indeed, if $X = Z(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$ is an affine variety in $\mathbb{A}^n$, then

$$\Delta(X) = Z(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n), x_1 - x_{n+1}, x_2 - x_{n+2}, \dots, x_n - x_{2n}),$$

so is closed in $\mathbb{A}^{2n}$.

Example 3.7. An example of a prevariety which is not a variety is given by the line with the doubled origin: this is obtained when we take two copies of $\mathbb{A}^1_k$, and glue them together using the identity map, but only along the open subset $\mathbb{A}^1_k \smallsetminus (0)$. All the points are identified except (0), so we obtain a space with two origins, which is a prevariety but not a variety. In the usual real or complex topology, this satisfies the conditions to be a manifold except that it is not Hausdorff.

Definition 3.8. A **subvariety** of a variety X is a subprevariety.

The terminology is justified by the following.

Proposition 3.9. *Let X be a variety, and $Y \subseteq X$ a subprevariety. Then Y is a variety.*

Proof. We have

$$\Delta(Y) = \Delta(X) \cap (Y \times Y) \subseteq X \times X.$$

Since $\Delta(X)$ is closed and $Y \times Y$ has the subset topology in $X \times X$, we conclude $\Delta(Y)$ is closed in $Y \times Y$. □

The following will be useful for checking that a prevariety is a variety.

Proposition 3.10. *A prevariety X is a variety if and only if for any two points $P, Q \in X$ there is an open subset U of X which contains P and Q and is a variety.*

Proof. If X is a variety, we can take the open subset to be all of X. Conversely, suppose the condition holds; we wish to show $\Delta(X)$ is closed. Thus, suppose (P, Q) is in the closure of $\Delta(X)$. By hypothesis, we can choose U containing P and Q and such that $\Delta(U)$ is closed in $U \times U$. Since $U \times U$ has the subset topology in $X \times X$, and $(P, Q) \in U \times U$, the hypothesis that (P, Q) is in the closure of $\Delta(X)$ implies it is in the closure of $\Delta(U)$, thus in $\Delta(U) \subseteq \Delta(X)$, and since P and Q were arbitrary, we conclude $\Delta(X)$ is closed. □

Corollary 3.11. *Any classical variety is a variety.*

Proof. We first claim that $\mathbb{P}^n$ is a variety. Although not every pair of points can be put into $\mathbb{P}^n \smallsetminus Z(x_i)$ for some i, for any pair of points there is a some hyperplane $H = Z(\sum_i c_i x_i)$ not containing either point, and $\mathbb{P}^n \smallsetminus H$ is still isomorphic to $\mathbb{A}^n$. The claim then follow from the previous proposition, since we already know any affine variety is a variety. The corollary then follows, since every classical variety is a subvariety of $\mathbb{P}^n$. □

Remark 3.12. Not every variety is classical. However, it is not so easy to write down examples of non-classical varieties. There are none in dimension 1. In dimension 2, there are no nonsingular examples, and in dimension 3 one obtains the first nonsingular variety which is not quasi-projective (see Example 3.4.1 of Appendix B of [1]). One can also produce examples of non-classical varieties among toric varieties associated to appropriately chosen fans.

4. Morphisms to affine and projective varieties

We conclude with a nagging loose end, explaining how morphisms with affine or projective target may be understood concretely in terms of tuples of regular functions. One may think of these regular functions as being given locally by polynomials, but using regular functions gives us the cleanest statements. We first observe that we can define $K(X)$ and $\mathcal{O}_{X,P}$ for prevarieties exactly as we did for classical varieties.

Suppose X is a prevariety, and $Y \subseteq \mathbb{A}^n$ an affine variety. Then we see that a function $X \to Y$ is equivalent to an n-tuple of functions $X \to k$, such that the induced map $X \to \mathbb{A}^n$ factors through Y. We can effectively use this correspondence to describe morphisms in terms of n-tuples of regular functions.

Proposition 4.1. *Given a map $\varphi : X \to Y$, with X any prevariety and $Y \subseteq \mathbb{A}^n$ an affine variety, φ is a morphism if and only if it is given by an n-tuple of regular functions on X.*

Proof. Certainly, if φ is a morphism, then pulling back the coordinate functions $x_1, \dots, x_n$ on $\mathbb{A}^n$ gives an n-tuple of regular functions on X, which describe φ. Conversely, suppose the pullbacks of the x_i are regular functions $f_i \in \mathcal{O}(X)$, so we have for any j that $f_i \circ \varphi_j$ is regular on X_j, using our standard atlas notation. But X_j is affine, so this means that the induced map $X_j \to Y$ is a morphism in the classical sense, and (using the one-chart atlas for Y) by Exercise 2.15 we conclude that φ is a morphism. □

We do not have such an elegant global description when Y is projective. We do see that if we have an $(n+1)$-tuple of functions $X \to k$, we get a function $X \dashrightarrow \mathbb{P}^n$ defined everywhere except where all $n+1$ functions vanish simultaneously, and conversely, any function $X \to \mathbb{P}^n$ can be written as an $(n+1)$-tuple of functions $X \to k$ which don't simultaneously vanish. However, this $(n+1)$-tuple is only defined up to simultaneous scaling by functions on X, and it is for this reason that when we restrict to morphisms and regular functions, the description does not globalize as it did in the affine case. However, we can still give a useful local characterization of morphisms.

Proposition 4.2. *Given a map $\varphi : X \to Y$, with X any prevariety and $Y \subseteq \mathbb{P}^n$ a projective variety., φ is a morphism if and only if it is described locally by $(n+1)$-tuples of regular functions which do not all vanish simultaneously.*

We first prove the following general lemma:

Lemma 4.3. *Let X be a prevariety, U an open subset, and $f : U \to k$ regular. Then f is a unit in $\mathcal{O}(U)$ if and only if $f(P) \neq 0$ for all $P \in U$.*

Equivalently, inside $K(X)$ we have

$$\mathcal{O}(U) = \bigcap_{P \in U} \mathcal{O}_{P,X}.$$

Proof. We first verify that the two statements are equivalent. Given $f \in \mathcal{O}(U)$, since $\mathcal{O}(U)$ injects into $K(X)$ we have that f is a unit in $\mathcal{O}(U)$ if and only if it is a unit in $\mathcal{O}(U)$ considered as a subring of $K(X)$. On the other hand, f is a unit in $\mathcal{O}_{P,X}$ if and only if it is not in the maximal ideal, if and only if $f(P) \neq 0$. Thus f has $f(P) \neq 0$ for all $P \in U$ if and only if $\frac{1}{f} \in K(X)$ is in $\mathcal{O}_{P,X}$ for all $P \in U$, which gives the desired equivalence.

But now the second statement is trivially verified: we certainly have $\mathcal{O}(U) \subseteq \bigcap_{P \in U} \mathcal{O}_{P,X}$, and if we have $f \in \bigcap_{P \in U} \mathcal{O}_{P,X}$, then by definition for all $P \in U$ we have representatives for f of the form $\langle U_P, f_P \rangle$ where $P \in U_P$, and $f_P \in \mathcal{O}(U_P)$. Moreover, since these all represent f we have $f_P|_{U_P \cap U_Q} = f_Q|_{U_P \cap U_Q}$ for all $P, Q \in U$. We now define f as a regular function on U by setting $f(P) = f_P(P)$; by the above, for any P we in fact have $f(Q) = f_P(Q)$ for all $Q \in U_P$, so since regularity is a local condition and the U_P cover U, we have constructed f as a regular function on U, as desired. □

Proof of Proposition 4.2. First suppose φ is a morphism, and let $V_j = Y \smallsetminus Z(x_j)$ for $j = 0, \dots, n$. Then on each V_j we have the regular functions induced by $\frac{x_i}{x_j}$ on $\mathbb{P}^n$ for $i = 0, \dots, n$, which gives us an $(n+1)$-tuple of regular functions on $\varphi^{-1}(V_j)$ by composing with φ. Noting that in $\mathbb{P}^n \smallsetminus Z(x_j)$ the point $(c_0, \dots, c_n)$ is also represented by $(\frac{c_0}{c_j}, \dots, \frac{c_n}{c_j})$, and that $\frac{x_i}{x_i} = 1$ vanishes nowhere on $\varphi^{-1}(V_j)$, we see that φ is represented by this $(n+1)$-tuple on $\varphi^{-1}(V_j)$. Letting j vary, we find that φ is everywhere locally represented by $(n+1)$-tuples of regular functions which do not all vanish simultaneously.

Conversely, suppose that we have some open cover U_i of X such that on each U_i, we can express φ as an $(n+1)$-tuple of regular functions which do not all vanish simultaneously. By refining the U_i, we may assume they are affine. Since being a morphism is a local condition on X, it is enough to see that if we consider each U_i as a prevariety, then φ induces a morphism $U_i \to Y$. Moreover, since $U_i \to Y$ is a morphism if and only if the composed map $U_i \to \mathbb{P}^n$ is a morphism, it suffices to treat the case $Y = \mathbb{P}^n$. Thus, we have reduced to the case that X is affine, $Y = \mathbb{P}^n$, and φ is given globally by an $(n+1)$-tuple of regular functions $f_0, \dots, f_n$ on X. But if for $j = 0, \dots, n$ we write $U_j := X \smallsetminus Z(f_j)$, then by hypothesis the U_j cover X, and each $U_j = \varphi^{-1}(\mathbb{P}^n \smallsetminus Z(x_j))$. Taking coordinates $\frac{x_i}{x_j}$ (with $i \neq j$) on $\mathbb{P}^n \smallsetminus Z(x_j)$, the map $U_j \to \mathbb{P}^n \smallsetminus Z(x_j)$ is then given by the n-tuple of functions $\frac{f_i}{f_j}$ for $i \neq j$, which are regular by Lemma 4.3. Then our map $U_j \to \mathbb{P}^n \smallsetminus Z(x_j)$ is a morphism for every j by Proposition 4.1, and because being a morphism is local on the target, we conclude that φ is a morphism. □

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.