MATH 250C: PROBLEM SET #4
DUE 5/2/2008

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For the next two exercises, you will want to look at the arguments in the notes on profinite groups.

Exercise 1. Show that the following are equivalent for a profinite group G:

a) Every finite index subgroup of G is open;
b) $G \to \hat{G}$ is an isomorphism of underlying groups, ignoring the topology;
c) $G \to \hat{G}$ is an isomorphism of topological groups, where $\hat{G}$ is still constructed ignoring the topology on G.

Show also that under any of these conditions, we also have that G has a unique profinite (i.e., compact Hausdorff and totally disconnected) topology on its underlying group structure.

Exercise 2. One can define the profinite completion in the category of topological groups by taking the inverse limit over all quotients by open normal subgroups of finite index. This generalizes the usual profinite completion if one thinks of a group as a topological group with the discrete topology, and fixes the potential problems with the profinite completion of a profinite group.

(a) Show that the map $G \to \hat{G}$ is continuous.

(b) Show that this construction has the universal property that if $f : G \to H$ is a (continuous) homomorphism of topological groups, with H profinite, then f factors uniquely as $G \to \hat{G} \to H$ (where $G \to \hat{G}$ is the usual map, and $\hat{G} \to H$ is a continuous homomorphism).

(c) Show that G is profinite if and only if $G \to \hat{G}$ is an isomorphism.

Exercise 3. Let F be a field, and K/F an extension of degree n. Show that there is an injective homomorphism $K \to M_{n,n}(F)$, where $M_{n,n}(F)$ is the ring of $n \times n$ matrices over F. Thus $M_{n,n}(F)$ contains isomorphic copies of every extension of F of degree n.