

MATH 250C: PROBLEM SET #5
DUE 5/9/2008

BRIAN OSSERMAN

Exercise 1. Suppose F is a field of characteristic $\neq 2$. An extension K/F is **biquadratic** if $K = F(\sqrt{m}, \sqrt{n})$ for some $m, n \in F$, and $[K : F] = 4$. Show that if m, n are not perfect squares in F , then $F(\sqrt{m}, \sqrt{n})$ is biquadratic if and only if mn is not a perfect square in F . Show further that if $F(\sqrt{m}, \sqrt{n})$ is biquadratic, then $F(\sqrt{m}, \sqrt{n}) = F(\sqrt{m} + \sqrt{n})$.

Exercise 2. Suppose F is a field of characteristic $\neq 2$, and $a, b \in F$ with b not a square. Show that $\sqrt{a + \sqrt{b}} = \sqrt{m} + \sqrt{n}$ for some $m, n \in F$ if and only if $a^2 - b$ is a square in F .

When is $F(\sqrt{a + \sqrt{b}})$ a biquadratic extension of F ?

Exercise 3. A field F is **formally real** if -1 cannot be written as a sum of (any number of) squares in F . Suppose F is formally real, and $f(x) \in F[x]$ an irreducible polynomial of odd degree. Letting α be a root of $f(x)$ in some extension of F , show that $F(\alpha)$ is still formally real. (Hint: consider a counterexample of minimal degree, and then use the definitions to produce a counterexample of smaller degree)

Exercise 4. Let K/F be an algebraic field extension, and suppose R is any subring of K containing F . Show that R is in fact a field.

Exercise 5. Suppose we have extensions K/F and E/F , both contained in some larger field.

(a) Show that if K/F is normal, then EK/E is normal.

(b) Show that if K/F and E/F are both normal, then KE/F is normal.