# MATH 250C: PROBLEM SET #7
# DUE 5/23/2008

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*Exercise* 1. Determine the Galois group of the splitting field $K$ of the polynomial $(x^2 - 2)(x^2 - 3)(x^2 - 5)$ over $\mathbb{Q}$. Find all fields lying between $K$ and $\mathbb{Q}$.

*Exercise* 2. Suppose $K/F$ is a Galois extension of degree $p^n$ for some prime $p$, and $n \geq 1$. Show that there are Galois extensions of $F$ contained in $K$ with degrees $p$ and $p^{n-1}$ over $F$.

*Exercise* 3. Suppose $f(x) \in \mathbb{Q}[x]$ is irreducible of degree 4, and the splitting field $K$ of $f(x)$ over $\mathbb{Q}$ has $\mathrm{Gal}(K/\mathbb{Q}) \cong S_4$ (there are many such polynomials). Let $\alpha \in K$ be a root of $f(x)$; show that $\mathbb{Q}(\alpha)$ has degree 4 over $\mathbb{Q}$, and there are no fields lying strictly between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$.

On the other hand, show that if $E/L$ is Galois of degree 4, then there are fields lying strictly between $L$ and $E$.

*Exercise* 4. Suppose $F$ has characteristic $p$, and suppose $n$ is a multiple of $p$. Write $n = p^k m$, with $p$ not dividing $m$. Show that the $n$th roots of unity in $F$ are precisely the same as the $m$th roots of unity in $F$.

*Exercise* 5. Let $K$ be a **number field**: that is, a finite extension of $\mathbb{Q}$. Show that $K$ contains only finitely many roots of unity.