

FREE PRODUCTS AND AMALGAMATED PRODUCTS

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As an application of free groups, we construct the free product and amalgamated product in group theory (these are coproducts in the modern category terminology).

1. FREE PRODUCTS

We suppose we are given a set S , and a collection of groups $\{G_s\}_{s \in S}$. The idea of the free product of the G_s is to take the union of the generators and relations for the G_s , with no additional relations. The free product thus contains copies of the G_s as disjoint subgroups, and in fact we will see that it satisfies a universal property which shows that it is independent of the choice of presentations for the G_s . Free products arise in algebraic topology when comparing the fundamental groups of two (or more) topological spaces to the fundamental group of the space obtained by joining them at a point.

Before formalizing the idea of building the free product from presentations, we observe that the construction of free groups via words immediately implies:

Lemma 1.1. *Given a set injection $S \hookrightarrow S'$, the induced homomorphism $F(S) \rightarrow F(S')$ is injective.*

We then make the following definition:

Definition 1.2. The **free product** $\coprod_{s \in S} G_s$ is constructed as follows: choose presentations (T_s, R_s) for each G_s , and let T be the disjoint union of the T_s , considered as sets. For each s , we then have an injective homomorphism $F(T_s) \hookrightarrow F(T)$ induced by $T_s \hookrightarrow T$, and let R be the union of the images in $F(T)$ of the sets $R_s \subseteq F(T_s)$. Let $\langle R \rangle$ be the normal subgroup of $F(T)$ generated by R . Then we set

$$\coprod_{s \in S} G_s := F(T) / \langle R \rangle,$$

with the injections $G_s \hookrightarrow \coprod_{s \in S} G_s$ determined by $F(T_s) \hookrightarrow F(T)$ and $F(T_s) \rightarrow G_s$.

When S is finite or countably infinite, the free product is frequently written with a $*$: for instance, $G_1 * G_2$.

A priori, this construction of $\coprod_{s \in S} G_s$ depends on the choice of presentations of the G_s , but we now show that in fact the group $\coprod_{s \in S} G_s$ together with the injections $G_s \hookrightarrow \coprod_{s \in S} G_s$ is uniquely determined by a universal property:

Proposition 1.3. *For any group G' , the natural map induced by composing with each $G_s \hookrightarrow \coprod_{s \in S} G_s$ induces a bijection*

$$\mathrm{Hom}\left(\coprod_{s \in S} G_s, G'\right) \xrightarrow{\sim} \prod_{s \in S} \mathrm{Hom}(G_s, G').$$

Proof. We use the construction of $\coprod_{s \in S} G_s$ to construct a map

$$\prod_{s \in S} \text{Hom}(G_s, G') \rightarrow \text{Hom}\left(\prod_{s \in S} G_s, G'\right)$$

inverting the map given by composition with $G_s \hookrightarrow \prod_{s \in S} G_s$.

Given $(\varphi_s : G_s \rightarrow G')_{s \in S}$, in the notation of the above definition, we obtain a family of homomorphisms $F(T_s) \rightarrow G'$ by composing with $F(T_s) \rightarrow G_s$, with kernel containing R_s . Since $T = \prod_{s \in S} T_s$, this family induces a homomorphism $F(T) \rightarrow G'$ which agrees with the given homomorphisms $F(T_s) \rightarrow G'$. Moreover, the kernel of the homomorphism contains the union of the images of R_s , and hence contains $\langle R \rangle$, meaning it gives a well-defined map $\prod_{s \in S} G_s = F(T)/\langle R \rangle \rightarrow G'$.

It is then a simple diagram chase to see that the constructed map indeed inverts the one induced by composition with $G_s \hookrightarrow \prod_{s \in S} G_s$, so we obtain the desired bijectivity statement. \square

This immediately implies that any two constructions of the free product, together with the inclusions of the G_s , differ by a unique isomorphism commuting with the inclusion maps.

Remark 1.4. Note that unlike the direct product, the free product of finite groups does not preserve finiteness. Indeed, the free product of two or more non-trivial groups is always infinite, because there are no relations described how non-trivial elements of the given groups commute with one another.

2. AMALGAMATED PRODUCTS

Free products of groups are generalized by a notion of amalgamated products of groups joined together along specified subgroups. For the sake of concreteness, we will carry out this construction for an amalgamated product of two groups. Suppose we have G_1, G_2 , and homomorphisms $f_1 : H \rightarrow G_1$, and $f_2 : H \rightarrow G_2$. We define:

Definition 2.1. The **amalgamated product** $G_1 *_H G_2$ is defined as follows: let N be the normal subgroup of $G_1 * G_2$ generated by elements of the form $f_1(h)f_2(h)^{-1}$ for $h \in H$; then

$$G_1 *_H G_2 := (G_1 * G_2)/N.$$

Note that $G_1 * G_2$ can be expressed as the special case of the amalgamated product where H is trivial. The amalgamated product satisfies a natural universal property generalizing the one for the free product:

Proposition 2.2. For a group G' , write $\text{Hom}(G_1, G') \times_H \text{Hom}(G_2, G')$ for

$$\{(g_1, g_2) \in \text{Hom}(G_1, G') \times \text{Hom}(G_2, G') : f_1 \circ g_1 = f_2 \circ g_2\}.$$

Then for any G' , the natural map induced by composition with $G_1 \rightarrow G_1 *_H G_2$ and $G_2 \rightarrow G_1 *_H G_2$ induces a bijection

$$\text{Hom}(G_1 *_H G_2, G') \xrightarrow{\sim} \text{Hom}(G_1, G') \times_H \text{Hom}(G_2, G').$$

Proof. We have by construction that $\text{Hom}(G_1 *_H G_2, G')$ is the subgroup of $\text{Hom}(G_1 * G_2, G')$ consisting of homomorphisms vanishing on N , and also that $\text{Hom}(G_1, G') \times_H \text{Hom}(G_2, G')$ is a subgroup of $\text{Hom}(G_1, G') \times \text{Hom}(G_2, G')$. By our basic result on free products we already have a bijection between the larger groups, so it suffices to

check that under the known bijection, the image of $\text{Hom}(G_1 *_H G_2, G')$ is precisely $\text{Hom}(G_1, G') \times_H \text{Hom}(G_2, G')$.

An element $\varphi \in \text{Hom}(G_1 *_H G_2, G')$ lies in $\text{Hom}(G_1 *_H G_2, G')$ if and only if it vanishes on N , which is equivalent to the condition that for all $h \in H$, we have $\varphi(f_1(h)f_2(h)^{-1}) = 1$, or equivalently, $\varphi(f_1(h)) = \varphi(f_2(h))$. But this is precisely the condition that the pair of homomorphisms obtained by restriction to G_1 and G_2 is an element of $\text{Hom}(G_1, G') \times_H \text{Hom}(G_2, G')$, proving what we wanted to show. \square

The amalgamated product also arises naturally in topology: the fundamental group of the gluing of two topological spaces along given subspaces is the amalgamated product of the fundamental groups of the two spaces, over the fundamental group of the subspaces being glued.