MATH 254A PROBLEM SET 6
DUE MONDAY, OCT. 31

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For these exercises, use that if $(a,b)=1$, $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} \cong \mathbb{Z}/ab\mathbb{Z}$.

Exercise 1. Show that if χ, ψ are Dirichlet characters of conductors f_χ, f_ψ, and $(f_\chi, f_\psi) = 1$, then the conductor of $\chi \cdot \psi$ is $f_\chi f_\psi$.

Exercise 2. Show that if $f = \prod_{i=1}^n p_i^{e_i}$, and χ is a character of conductor f, then χ may be written uniquely as $\chi_{p_1} \cdots \chi_{p_n}$, where χ_{p_i} is a character of conductor $p_i^{e_i}$.

Exercise 3. A Dirichlet character χ is called **quadratic** if its square is χ_1, or equivalently, if χ takes values in ± 1. Show explicitly that there is a one-to-one correspondence between quadratic Dirichlet characters and quadratic fields as follows (note that you are not required relate this to the more general equivalence described in lecture between groups of Dirichlet characters modulo n and subfields of $\mathbb{Q}(\zeta_n)$):

a) Show that there is a unique quadratic character of conductor n when n is an odd prime or 4, and that there are exactly 2 quadratic characters of conductor 8, one odd and one even.
b) Show that if $D \in \mathbb{N}$ is the conductor of a quadratic character, then D must be of the form dm, with m odd and square-free, and $d = 1, 4, 8$.
c) Show the same for $D \in \mathbb{N}$ with $D = |D_K|$ for K a quadratic field.
d) With $D = dm$ as above, show that if $d < 8$, there exists a unique quadratic character of conductor D and a unique quadratic field K with $|D_K| = D$, and note that the character is even if and only if K is real.
e) With $D = dm$ as above, if $d = 8$ show that there exist exactly 2 quadratic characters of conductor D, one odd and one even, and exactly 2 quadratic fields K with $|D_K| = D$, one imaginary and one real.