MATH 254A PROBLEM SET 7
DUE MONDAY, NOV. 7

BRIAN OSSERMAN

In these exercises, for p an odd prime, and $j > 0$, $\left(\frac{j}{p}\right)$ denotes the Legendre symbol, defined to be 1 if j is a quadratic residue modulo p and -1 otherwise.

Exercise 1. Show that if $p \equiv 1 \pmod 4$, that

$$\prod_{0<j<\frac{p}{2}:\left(\frac{j}{p}\right)=-1} \left(\sin \frac{\pi j}{p}\right) > \prod_{0<j<\frac{p}{2}:\left(\frac{j}{p}\right)=1} \left(\sin \frac{\pi j}{p}\right).$$

Note that the number of quadratic residues and non-residues in $(0, \frac{p}{2})$ is the same. Since sin is monotone increasing in $[0, \frac{\pi}{2}]$, this exercise shows that quadratic residues cluster in the first half of the interval $(0, \frac{p}{2})$.

Exercise 2. Show that for $p \equiv 3 \pmod 4$, and $p > 3$, the class number formula for $\mathbb{Q}(\sqrt{-p})$ may be rewritten as

$$h_K = \frac{1}{2 - \left(\frac{2}{p}\right)} \sum_{0<j<\frac{p}{2}} \left(\frac{j}{p}\right),$$

and conclude that there are more quadratic residues in the interval $(0, \frac{p}{2})$ than non-quadratic residues (of course, this conclusion holds also for $p = 3$).

Hint: use the behavior of $\left(\frac{j}{p}\right)$ under $j \mapsto p - j$, and compare the sums obtained by separating j first by size, and then by parity.