MATH 254A PROBLEM SET 2
DUE MONDAY, SEPT. 19

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This homework assignment is mainly intended to familiarize people with the commutative algebra ideas we have used to study Dedekind domains. With the exception of 3, the problems are therefore very routine.

Exercise 1. Show that if one considers the map from ideals of $R_{\mathfrak{p}}$ to ideals of R given by $I \mapsto I \cap R$, this map is injective, and if restricted to prime ideals, gives a bijection between prime ideals of $R_{\mathfrak{p}}$ and prime ideals of R contained in $\mathfrak{p}$.

In particular, if R is Noetherian, then $R_{\mathfrak{p}}$ is Noetherian.

Exercise 2. Show that a ring R is Noetherian (i.e., every ascending chain of ideals stabilizes) if and only if every ideal is finitely generated.

Exercise 3. Show that if R is a Noetherian ring, and I an ideal, such that there exists a unique prime ideal $\mathfrak{p}$ containing I, then I contains some power of $\mathfrak{p}$.

Hint: first show that if an element x is contained in every prime ideal of a ring, then $x^n = 0$ for some n. Do this by considering the family of all ideals such that no power of x lies in them. Next, consider the ring R/I.

Exercise 4. Show that if K is the field of fractions of a Noetherian integral domain R, and $I \subset K$ is an R-module (i.e., it is closed under addition, and under multiplication by elements of R), then I is a fractional ideal of R if and only if I is finitely generated.

Exercise 5. Show that for any integral domain R, if I and J are fractional ideals of R, then $(I : J)$ is a fractional ideal of R.

Exercise 6. Show that in any integral domain R, if a fractional ideal I has an inverse, it must be given by I'.

Check that if R is a PID, then I' is the inverse of I.

Exercise 7. Check that for fractional ideals I, J, and a prime ideal $\mathfrak{p}$, we have $I_{\mathfrak{p}} J_{\mathfrak{p}} = (IJ)_{\mathfrak{p}}$, and if R is Noetherian, $(I_{\mathfrak{p}} : J_{\mathfrak{p}}) = (I : J)_{\mathfrak{p}}$.

The last two exercises are examples showing that the results we have proved for Dedekind domains typically do not hold for more general rings. For the first part of each exercise, it helps to consider the ring obtained by modding out by the ideal given.

Exercise 8. Show that in $\mathbb{Z}[\sqrt{-3}]$, the ideal generated by 2 cannot be written as a product of prime ideals. Show that the ideal $(2, 1 + \sqrt{-3})$ does not have any fractional ideal inverse.

Exercise 9. Show that in $k[x, y]$, where k is any field, the ideal (x, y^2) cannot be written as a product of prime ideals. Show that (x, y) does not have any fractional ideal inverse.