MATH 254A PROBLEM SET 4
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Exercise 1. Let R be an integral domain with field of fractions K, and L some extension of K. Show that R is integrally closed (in K) if and only if for all $\alpha \in L$ such that α is integral over R, the monic minimal polynomial for α over K has coefficients in R.

Exercise 2. a) For all square-free integers n, let $K = \mathbb{Q}(\sqrt{-n})$, and compute D_K, and then carefully and completely describe how every prime $p \in \mathbb{Z}$ factors in $\mathscr{O}_K$.

b) Show that for p prime to $2n$, we have $p\mathbb{Z}[\sqrt{-n}] = \mathfrak{p}_1\mathfrak{p}_2$ if and only if $-n$ has a square root mod p.

Exercise 3. Fix $n \in \mathbb{N}$: describe which primes $p \in \mathbb{Z}$ not dividing n **split completely** in $\mathscr{O}_{\mathbb{Q}(\zeta_n)}$: that is, which p factor into $\phi(n)$ distinct (prime ideal) factors, each with norm p, in $\mathscr{O}_{\mathbb{Q}(\zeta_n)}$.

It is a remarkable consequence of the density theorems that a number field is completely determined by which primes split completely in it. However, the proof of the following special case of the Dirichlet density theorem is considerably easier:

Exercise 4. Show that for any n, there are infinitely many primes congruent to 1 mod n.

Hint: argue by contradiction, and evaluate $\Phi_n(x)$ at large, cleverly chosen values.