MATH 256A: PROBLEM SET #1
DUE 9/7/2006

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1. On sheaves (and Grassmannians, and functors...)

First, some exercises to familiarize yourself with sheaves, as well as with the basic language of categories and functors.

The first exercise is intended to motivate constructing spaces locally, by the example of the Grassmannian. Let k be a field. Recall that the **Grassmannian** $G(r,d)$ parametrizes r-dimensional subspaces of a fixed d-dimensional vector space V.

Exercise 1. Show that $G(r,d)$ can be covered by "open" subsets (we haven't defined a topology yet), each of which is isomorphic to (i.e., in bijection with) $\mathbb{A}_k^{r(d-r)}$.

Hint: for each $W \subseteq V$ in $G(r,d)$, fix a $V_W \subseteq V$ of dimension $d-r$, and intersecting W only at the origin. Consider the "open neighborhood" of W consisting of spaces W' which intersect V_W only at the origin, and identify the neighborhood with $\text{Hom}(W, V_W)$, using that every vector in V can be written uniquely as a vector in W plus a vector V_W.

Recall that a **category** $\mathcal{C}$ consists of a collection of **objects** $\text{Obj}(\mathcal{C})$ and **morphisms** between objects: i.e., for each ordered pair (a,b) of (not necessarily distinct) objects of $\mathcal{C}$, we have a set $\text{Mor}(a,b)$ of morphisms from a to b. We are also given the data of **composition** of morphisms: for any (a,b,c) of objects on $\mathcal{C}$, a map of sets $\text{Mor}(a,b) \times \text{Mor}(b,c) \to \text{Mor}(a,c)$; given $f \in \text{Mor}(a,b)$ and $g \in \text{Mor}(b,c)$, we write the resulting element of $\text{Mor}(a,c)$ with the usual notation $g \circ f$.

The two conditions for such a collection of data to form a category are:

(i) that composition is **associative**, i.e. that for all (a,b,c,d) and all f,g,h in $\text{Mor}(a,b)$, $\text{Mor}(b,c)$ and $\text{Mor}(c,d)$ respectively, that $h \circ (g \circ f) = (h \circ g) \circ f$;

(ii) that for any object $a \in \text{Obj}(\mathcal{C})$, there is an **identity** element $1_a \in \text{Mor}(a,a)$ such that for any b and any morphism $f \in \text{Mor}(a,b)$, we have $f \circ 1_a = f$, and for any $f \in \text{Mor}(b,a)$ we have $1_a \circ f = 1_a$.

For instance, we have the category of sets, where the objects are sets, and the morphisms are arbitrary functions, the category of topological spaces, where the objects are topological spaces and the morphisms are continous functions, and the category of rings, where the objects are rings and the morphisms are homomorphisms.

We also have the notion of a **functor**. A functor can be either **covariant** or **contravariant**. A covariant functor is a mapping between categories; specifically, $F: \mathcal{C}_1 \to \mathcal{C}_2$ associates to each object a of $\mathcal{C}_1$ an object $F(a)$ of $\mathcal{C}_2$, and associates, for any pair (a,b) of objects of $\mathcal{C}_1$ and any morphism $f \in \text{Mor}(a,b)$, a morphism $F(f) \in \text{Mor}(f(a), f(b))$. A functor must have the properties that $F(1_a) = 1_{F(a)}$ for any $a \in \text{Obj}(\mathcal{C}_1)$, and that for any (a,b,c) and $f \in \text{Mor}(a,b), g \in \text{Mor}(b,c)$, we have $F(g \circ f) = F(g) \circ F(f)$. A contravariant functor is the same, except that it reverses directions of morphisms, so that if $f \in \text{Mor}(a,b)$, then $F(f) \in \text{Mor}(F(b), F(a))$.

Exercise 2. a) Show that given a topological space X, we can consider the topology on X as a category $\mathcal{C}_X$ whose objects are the open subsets of X, and whose morphisms are the inclusion mappings.

b) Show that a presheaf of rings on X is the same as a contravariant functor from $\mathcal{C}_X$ to the category of rings.

The following exercise requires the notions of stalks and locally ringed spaces (as well as the fact that $\operatorname{Spec} R$ is a locally ringed space), to be covered in lecture on 9/5.

Exercise 3. a) Do Hartshorne, Exercise 1.2 of Chapter II.

b) Show that the definition given in Hartshorne of a scheme is the same as the definition given in lecture (i.e., using ringed spaces instead of locally ringed spaces). Feel free to compliment my definition on its elegance and simplicity, but note that I will not be grading the problem sets, so doing so will not improve your grade.

2. On schemes

Several useful examples, and leading into some important concepts (residue fields, tangent spaces, K-valued points...)

The second part of the next exercise in principle requires the definition of a morphism of schemes, which will be covered in lecture on 9/5. However, you may use without proof the result of Exercise 2.4.

Exercise 4. Do Hartshorne, Exercise 2.5 of Chapter II.

Exercise 5. Do Hartshorne, Exercise 2.10 of Chapter II.

The following three exercises require the notion of local rings of schemes, to be covered in lecture on 9/5.

Exercise 6. Do Hartshorne, Exercise 2.7 of Chapter II.

Exercise 7. Do Hartshorne, Exercise 2.8 of Chapter II.

Exercise 8. Do Hartshorne, Exercise 2.11 of Chapter II.