

**MATH 256A: PROBLEM SET #11**  
**DUE 11/16/2006**

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The following exercise sounds complicated, but is a situation one frequently runs into in making arguments involving degenerations.

*Exercise 1.* Suppose we are given  $f : X \rightarrow Y$ , with  $X$  and  $Y$  Noetherian and irreducible,  $f$  open and of finite type, and  $X$  regular and  $Y$  universally catenary. Suppose further that every component of every fiber of  $f$  has some given dimension  $d$ . Given  $Z_1, \dots, Z_n$  closed subschemes of  $X$  of codimensions  $d_1, \dots, d_n$ , set  $\rho = d - \sum_i d_i$ , and suppose  $\rho \geq 0$ .

Set  $Z = \bigcap_i Z_i \subseteq X$ . Suppose we are given a point  $z \in Z$ , with image  $f(z) = y$  in  $Y$ . Show that every component of  $f^{-1}(y) \cap Z$  has dimension at least  $\rho$ . Suppose that  $f^{-1}(y) \cap Z$  has dimension exactly  $\rho$ . Show that every neighborhood  $U$  of  $z$  in  $Z$  maps dominantly to  $Y$ .

We give Nagata's example of an infinite-dimensional Noetherian ring.

*Exercise 2.* (a) Suppose that  $R$  is a ring such that: (i) for every maximal ideal  $\mathfrak{m}$ , we have  $R_{\mathfrak{m}}$  Noetherian; and (ii) every non-zero element  $r \in R$  is contained in only finitely many maximal ideals of  $R$ . Show that  $R$  is Noetherian.

(b) Now let  $R_0 = k[x_1, x_2, \dots]$  be the polynomial ring in infinitely many variables over a field  $k$ , and choose an increasing function  $d : \mathbb{N} \rightarrow \mathbb{N}$ . Define a sequence of prime ideals  $P_i$  of  $R_0$  by  $P_i = (x_{d(i)1}, \dots, x_{d(i+1)})$ , and set  $S$  to be the multiplicative set  $R_0 \setminus (\cup_{i=1}^{\infty} P_i)$ . Finally, let  $R = S^{-1}R_0$ . Show that the maximal ideals of  $R$  are the ideal  $S^{-1}P_i$ .

(c) Conclude that for an appropriate choice of  $d$ , the  $R$  of (b) is a Noetherian ring of infinite dimension.

The following exercise is intended to provide some practice with regular and local complete intersection rings, and cotangent spaces.

*Exercise 3.* (a) Let  $\varphi : A \twoheadrightarrow B$  be a surjection of Noetherian local rings, inducing an isomorphism on cotangent spaces, and suppose further that  $B$  is regular. Show that  $\varphi$  is an isomorphism. [Geometrically, this says that a regular scheme is as large as possible for its cotangent space.] Hint: you may use that a regular local ring is necessarily an integral domain.

(b) Give an example to show that (a) fails if we assume instead that  $A$  is regular.

(c) Use (a) to show that if  $X \subseteq Y$  is a closed subscheme, with both  $X$  and  $Y$  regular and irreducible, then  $X$  is necessarily a local complete intersection in  $Y$ .

*Exercise 4.* Do Hartshorne, Exercise 9.1 of Chapter III.

*Exercise 5.* Do Hartshorne, Exercise 9.3 of Chapter III.