MATH 256A: PROBLEM SET #12, PART 2
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We say that $(X, \mathcal{O}_X)$ is a **complex locally ringed space** if it is a locally ringed space such that every ring $\mathcal{O}_X(U)$ has the structure of a $\mathbb{C}$-algebra, in a way that is compatible with restriction maps. A **morphism** of complex locally ringed spaces is a morphism of locally ringed spaces respecting the $\mathbb{C}$-algebra structures.

Exercise 1. Let X be a scheme of finite type over $\operatorname{Spec} \mathbb{C}$, with associated analytic topological space X_{an}, and sheaf $\mathcal{O}_X^{\text{an}}$. Let A be a $\mathbb{C}$-algebra of finite type, with a unique maximal ideal (and hence necessarily an Artin local ring).

(a) Show that $\operatorname{Spec} A$ has the same sheaf as either a scheme or its associated analytic space.

(b) Show that morphisms of schemes $\operatorname{Spec} A \to X$ are in bijection with morphisms of complex locally ringed spaces $\operatorname{Spec} A \to (X_{\text{an}}, \mathcal{O}_X^{\text{an}})$. [See also Exercise 1 of Problem Set 2.]

We can make $\mathbb{C}^n$ into a complex locally ringed space simply by considering the sheaf of analytic functions. We say that a complex locally ringed space $(X, \mathcal{O}_X)$ is a **complex manifold** of dimension n if it is Hausdorff and locally isomorphic to (open subsets of) $\mathbb{C}^n$. [Note that this is equivalent to the more classical definitions in terms of an open cover with analytic transition functions.]

Exercise 2. Let X be a scheme of finite type over $\operatorname{Spec} \mathbb{C}$.

Show that X is smooth and separated over $\operatorname{Spec} \mathbb{C}$ if and only if $(X_{\text{an}}, \mathcal{O}_X^{\text{an}})$ is a complex manifold.

Exercise 3. Let X and Y be schemes, and $f : X \to Y$ a morphism.

(a) Given $x \in X$, write $y = f(x)$, and suppose that the residue fields $k(x)$ and $k(y)$ are isomorphic under the map induced by f. Show that f is étale at x if and only if f induces an isomorphism $\hat{\mathcal{O}}_{Y,y} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x}$.

(b) Now suppose that X and Y are of finite type over $\operatorname{Spec} \mathbb{C}$, and that x is a closed point. Show that f is étale at x if and only if f is a local isomorphism of complex locally ringed spaces $(X, \mathcal{O}_X^{\text{an}}) \to (Y, \mathcal{O}_Y^{\text{an}})$ in a neighborhood of x.