MATH 256A: PROBLEM SET #2
DUE 9/14/2006

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1. More on sheaves and structure sheaves

Exercise 1. a) Do Hartshorne, Exercise 1.18 of Chapter II.

b) Use this to show that if R is a ring with a unique prime ideal (such as k, or $k[\epsilon]/\epsilon^2$), and X any scheme, with point $x \in X$, then the morphisms $\operatorname{Spec} R \to X$ having image x are in one-to-one correspondence with local homomorphisms $\mathcal{O}_{X,x} \to R$.

Morally, we would like to define the structure sheaf $\mathcal{O}$ of an affine scheme $\operatorname{Spec} R$ by setting $\mathcal{O}(U) = \{\frac{a}{b} : a, b \in R, b \notin \mathfrak{q} \forall \mathfrak{q} \in U\}$; i.e., by starting with R and allowing division by any "function" not vanishing on U. This is a perfectly good presheaf, but it is not in general a sheaf:

Exercise 2. Let $R = k[x, y, z, w]/(xy - zw)$ for k an algebraically closed field. Show that the function $\frac{x}{z}$ on $U_1 = D(z)$ and the function $\frac{w}{y}$ on $U_2 = D(y)$ agree on $U_1 \cap U_2$, but cannot be written as $\frac{a}{b}$ for a, b with b not in $\mathfrak{q}$ for all $\mathfrak{q}$ in $U_1 \cup U_2$.

Exercise 3. Show that the structure sheaf of $\operatorname{Spec} R$ is the sheafification of the presheaf $\mathcal{O}$ defined above.

2. More on schemes

Exercise 4. Do Hartshorne, Exercise 2.1 of Chapter II.

Exercise 5. Do Hartshorne, Exercise 2.2 of Chapter II.

Exercise 6. a) Do Hartshorne, Exercise 2.4 of Chapter II.

b) For any given scheme X, describe the set of morphisms $X \to \operatorname{Spec} \mathbb{Z}[t]$.