MATH 256A: PROBLEM SET #4
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Exercise 1. (Gluing morphisms of schemes) Prove the following:

Let X and Y be schemes over S, and $\{U_i\}$ an open covering of X. Then morphisms $f : X \to Y$ over S are in one-to-one correspondence with collections of morphisms $f_i : U_i \to Y$ over S, such that for all i, j we have $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ as morphisms $U_i \cap U_j \to Y$ over S.

Exercise 2. (Gluing schemes) Do Hartshorne, Exercise 2.12 of Chapter II.

Exercise 3. Let S be a scheme, and $\mathcal{C}$ a half-full subcategory of Sch_S, such that if T is in $\mathcal{C}$, then every open subscheme of T is in $\mathcal{C}$. Given $X \in \mathrm{Sch}_S$, let $F : \mathcal{C} \to \mathrm{Set}$ be the functor $T \mapsto \mathrm{Mor}_{\mathrm{Sch}_S}(T, X)$ (that is, if X' represents F, then X' is universal for maps from $\mathcal{C}$ to X).

Show that F is a Zariski sheaf.

A scheme X is **integral** if for every U, we have $\mathcal{O}_X(U)$ is an integral domain. This is equivalent to X being irreducible and reduced; see p. 82 of Hartshorne. A morphism $f : X \to Y$ is **dominant** if $f(X)$ is dense in Y. If X is integral, we say that X is **normal** if for every $P \in X$, we have that the stalk $\mathcal{O}_{X,P}$ is (in addition to being an integral domain) integrally closed in its field of fractions.

Exercise 4. In the above exercise, suppose that X is integral, and let $\mathcal{C}$ be the category of normal schemes over S, with morphisms consisting only of dominant morphisms. Show that the F of the previous exercise is representable by first handling the case that X is affine, and then showing that considering an open affine cover of X gives a cover of F by open subfunctors.

This implies that the **normalization** of an integral scheme exists: i.e., there is a scheme $X' \to X$ such that X' is normal, and any dominant morphism from a normal scheme to X factors uniquely through X'.

Exercise 5. Let C_1 be the plane curve given by $y^2 = x^3 + x^2$, and C_2 by $y^2 = x^3$. What do the real points of C_1 and C_2 look like? What are the normalizations of C_1 and C_2?

Exercise 6. Do Hartshorne, Exercise 3.10 of Chapter II.

Exercise 7. Do Hartshorne, Exercise 3.15 of Chapter II.

The following is a warm-up for proving that the Grassmannian exists.

Exercise 8. (a) For a ring A, define $\mathbb{A}^n_A$ to be $\operatorname{Spec} A[x_1, \dots, x_n]$ (if A is a k-algebra, then $\mathbb{A}^n_A$ is just $\operatorname{Spec} A \times_{\operatorname{Spec} k} \mathbb{A}^n_k$, but in general $\mathbb{A}^n_A$ can't be written as a product with $\mathbb{A}^n_k$ for a single field k). Show that $\mathbb{A}^n_A$ (together with the tuple $(x_1, \dots, x_n)$) represents the functor $F : \mathrm{Sch}_{\operatorname{Spec} A} \to \mathrm{Set}$ defined by $F(T) := \mathcal{O}_T(T)^n$ (i.e., T maps to n-tuples of global sections of the structure sheaf of T).

(b) For a scheme S, consider the functor $F : \mathrm{Sch}_S \to \mathrm{Set}$ defined by $F(T) := \mathcal{O}_T(T)^n$. Show that this is representable by a scheme, which we denote $\mathbb{A}^n_S$, and call **affine n-space** over S. Hint: first show that F is a Zariski sheaf, and then show that an open subscheme of S gives an open subfunctor of F, and apply (a).

(c) Give an alternate proof of (b) by showing that the scheme $S \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}}$ represents F.